# Discrete Differential - Surfaces

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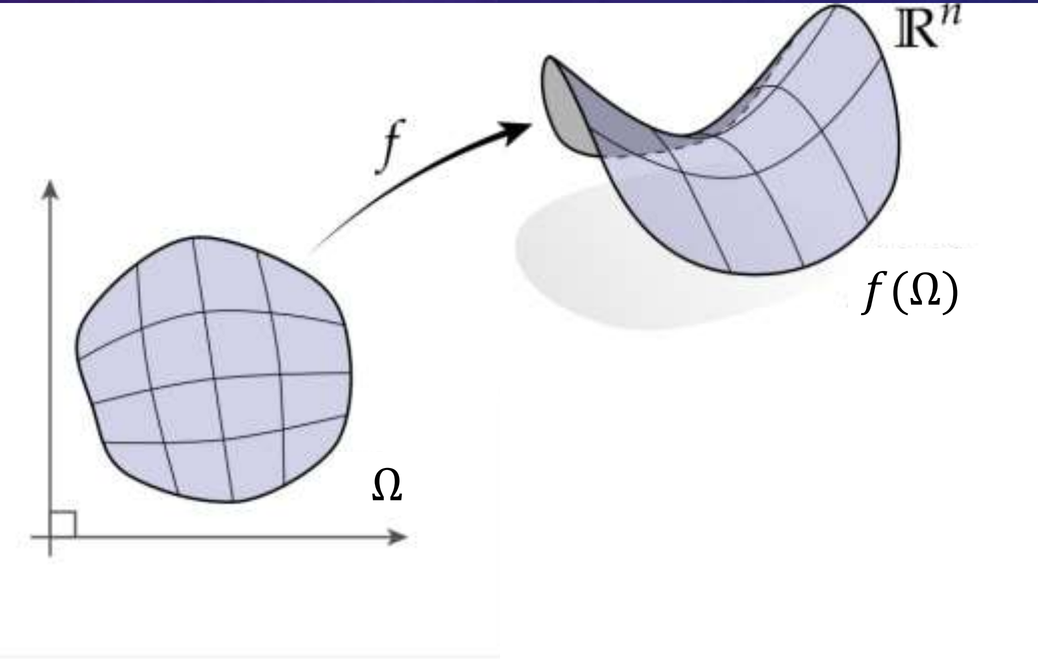
# Smooth surface

# Parameterized surface

- A parametrized surface is a continuous function $f \colon \Omega \to \mathbb{R}^3$, where the domain $\Omega \subseteq \mathbb{R}^2$ is some (connected) set on the plane.



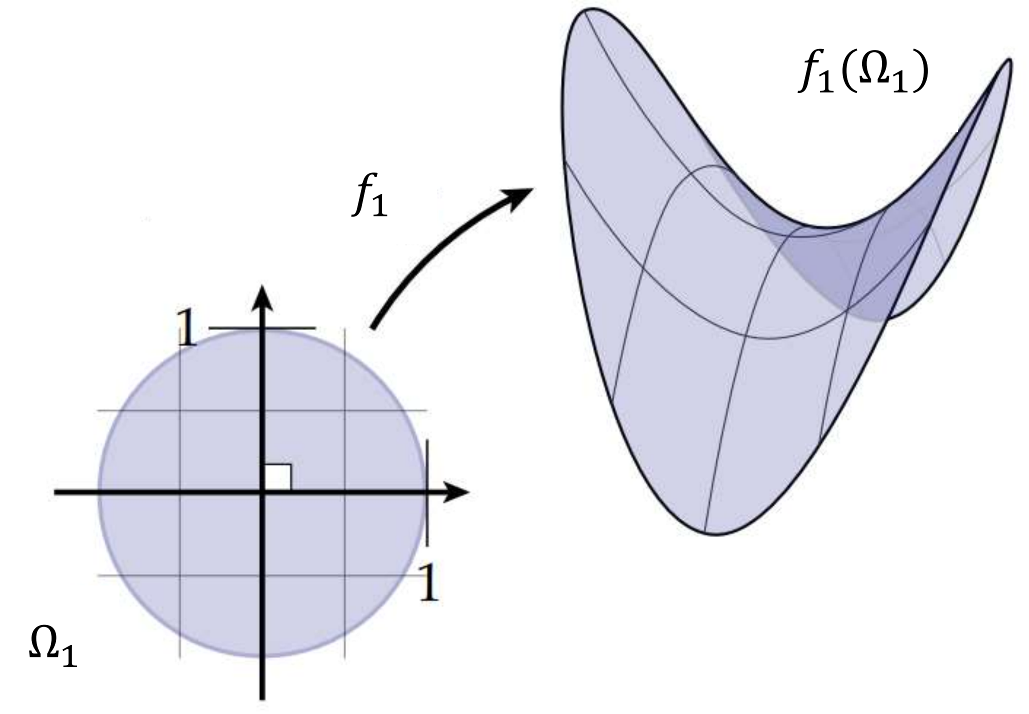
# Reparameterization

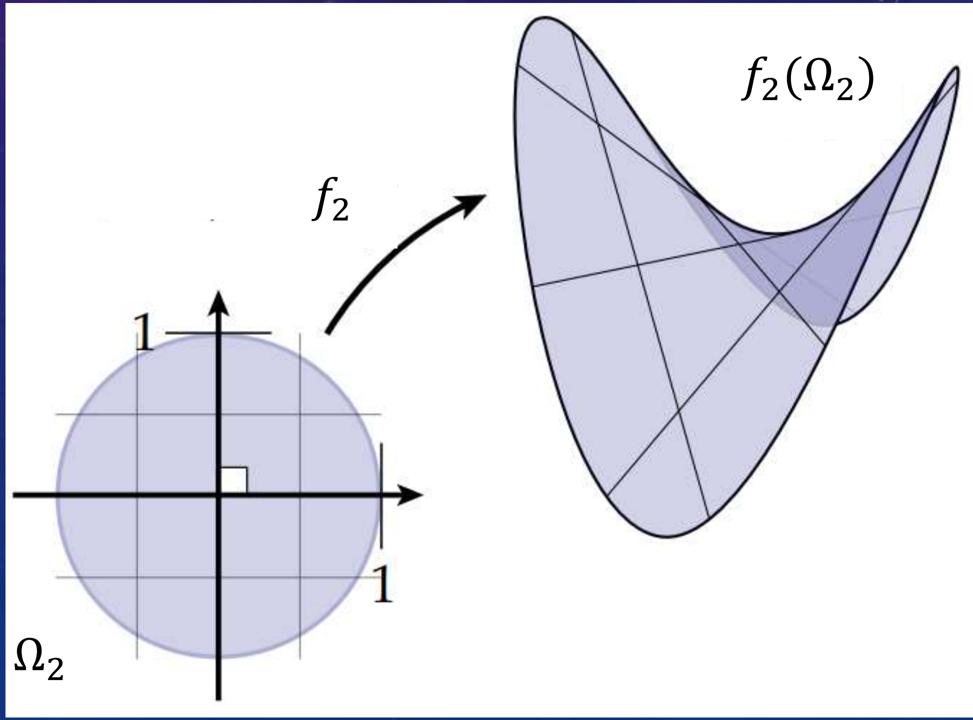
- Two parametrizations $f_1\colon \Omega_1 \to \mathbb{R}^3$and $f_2 : \Omega_2 \to \mathbb{R}^3$ are said to yield the same surface if there exists a continuous and continuously invertible function $\phi\colon \Omega_1 \to \Omega_2$ , called a reparameterization, such that $f_1(u, v) = f_2(\phi(u, v))$ for all $(\mathrm{u}, \mathrm{v}) \in \Omega_1$; in short $f_1 = f_2 \circ \phi$

# Reparameterization

$$\begin{cases} s \leftarrow \dfrac{\sqrt{2}}{2}(u+v) \\ t \leftarrow \dfrac{\sqrt{2}}{2}(u-v) \end{cases}$$

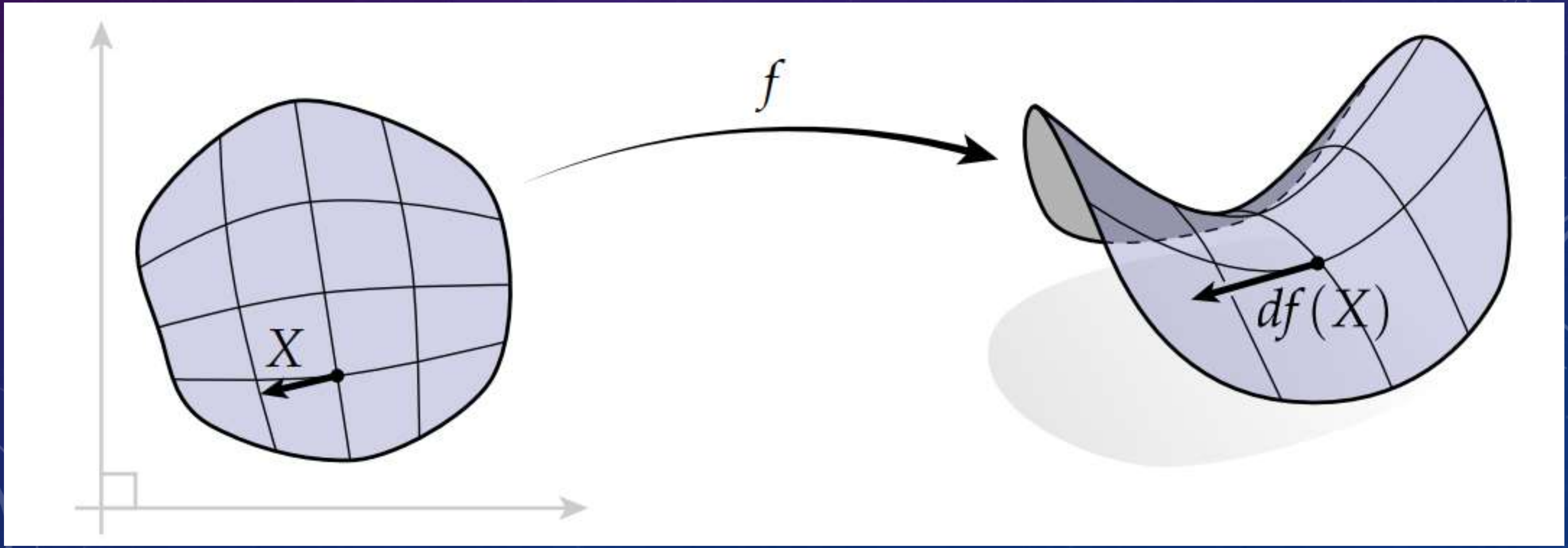
$$\Omega_1 = \Omega_2 = B_1(0), f_1(u,v) = (u, v, u^2 - v^2), f_2(s,t) = \left(\frac{\sqrt{2}}{2}(s+t), \frac{\sqrt{2}}{2}(s-t), 2st\right)$$



# Differential of a surface

- $df: X \in \Omega \to T_P f \in \mathbb{R}^3$ push forward $X$



# Differential in coordinates

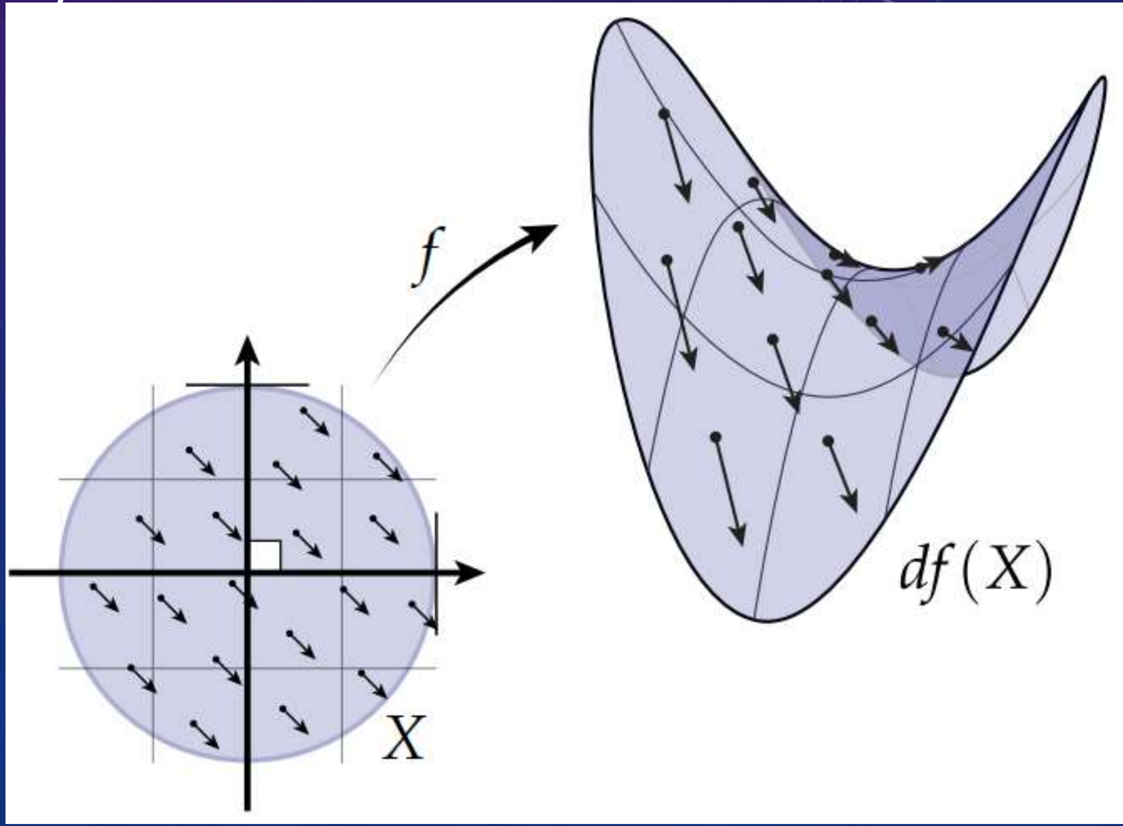
$$\Omega = \{u^2 + v^2 \le 1\}, f(u,v) = (u, v, u^2 - v^2)$$

$$df = \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv =$$

$$(1,0,2u)du + (0,-1,2v)dv$$

$$(u,v) = (0,0), (du, dv) = \frac{3}{4}(1,-1)$$

$$\Rightarrow df = (\tfrac{3}{4}, -\tfrac{3}{4}, 0)$$



# Differential – Jacobian matrix

Consider a map $f\colon \mathbb{R}^n \to \mathbb{R}^m$, let $(x_1, x_2, \dots, x_n)$ be the coordinates of $\mathbb{R}^n$. The Jacobian of $f$ is the matrix

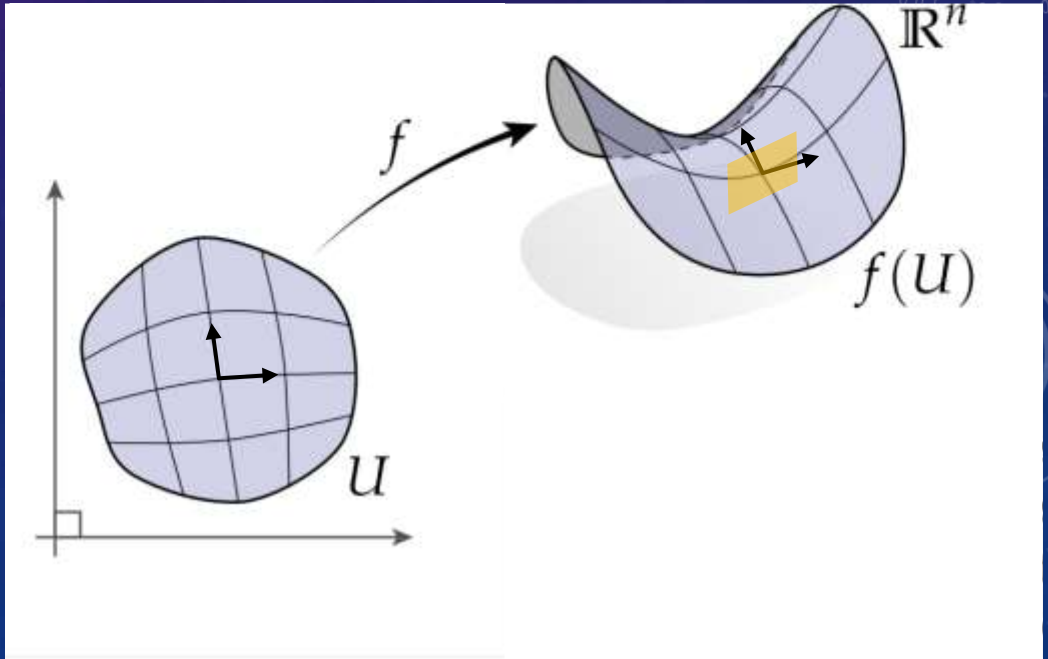
$$J_f = \begin{bmatrix} \frac{\partial f^1}{\partial x_1} & \cdots & \frac{\partial f^1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f^m}{\partial x_1} & \cdots & \frac{\partial f^m}{\partial x_n} \end{bmatrix}$$

where $f^1, f^2, \dots, f^m$ are the components of $f$. The differential in matrix representation are $df(X) = J_f X$.

# Tangent plane

Surface $f : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3, J_f$ 3 × 2 matrix and $df(X) = J_f X$.

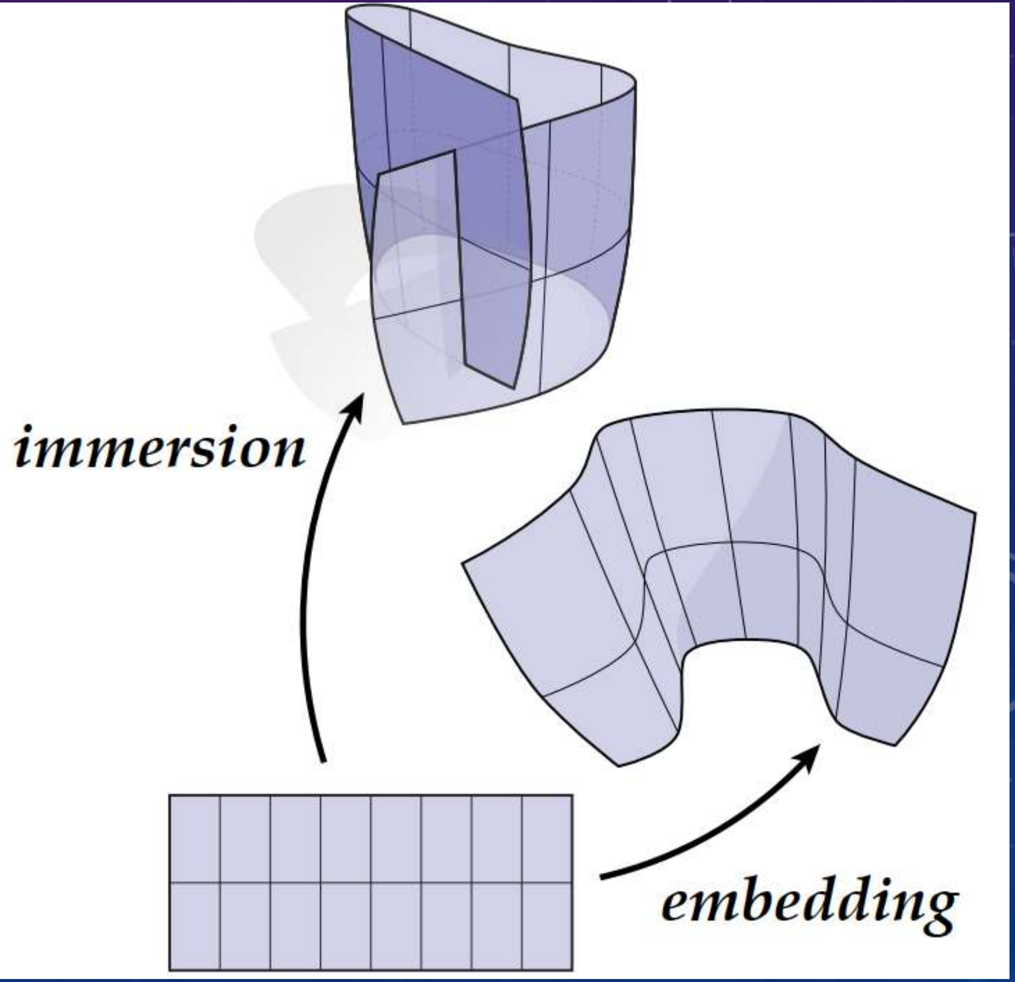
$$J_f X = \begin{bmatrix} \frac{\partial f^1}{\partial x_1} & \frac{\partial f^1}{\partial x_2} \\ \frac{\partial f^2}{\partial x_1} & \frac{\partial f^2}{\partial x_2} \\ \frac{\partial f^3}{\partial x_1} & \frac{\partial f^3}{\partial x_2} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} = [J_{e_1}\ J_{e_2}] \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix}$$




Normal : $J_f^{\perp} = \{J_n | J_n \perp J_{e_1}, J_n \perp J_{e_2}\}$

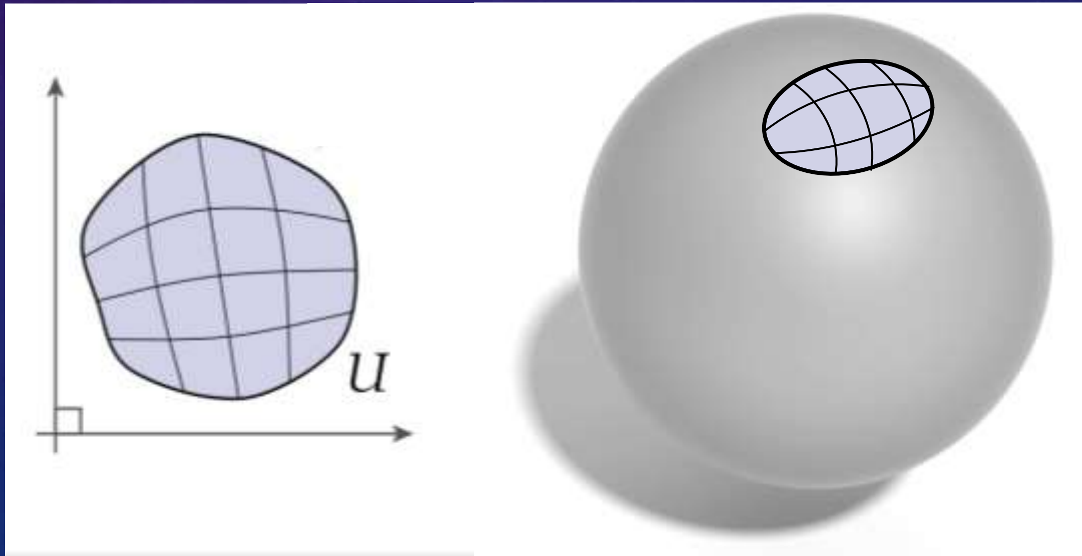
# Regular surface

- A parametrized surface $f : \Omega \rightarrow \mathbb{R}^3$ is regular (immersion) if $\gamma$ has continuous Jacobian, and has non-vanishing determinant $\left|J_f\right| \neq 0$ for every point.
- Immersion vs. embedding



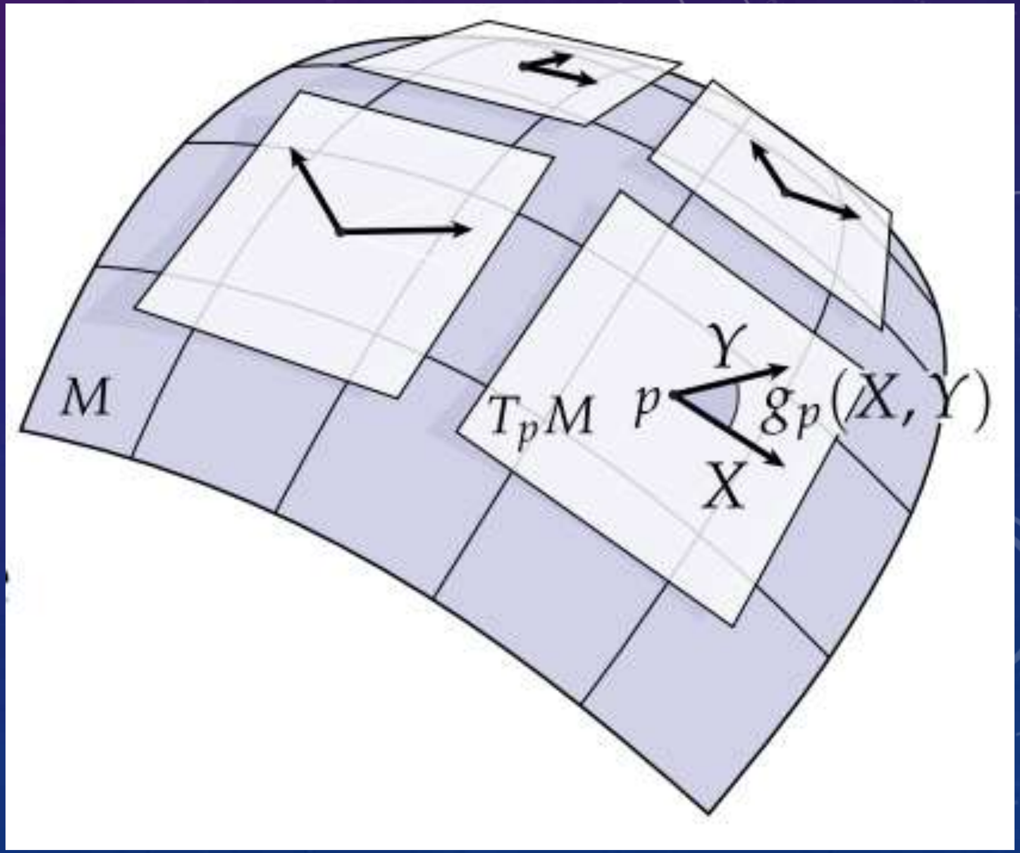
# Isometric parameterization

- For regular curves, reparametrized by arclength → isometric.
- For regular surfaces, things are different : metric and curvature



# Riemann metric

- Measurements of lengths and angles of tangent vectors $X, Y$
- This information is encoded by the so-called Riemannian metric $g(X, Y)$



# Riemann metric

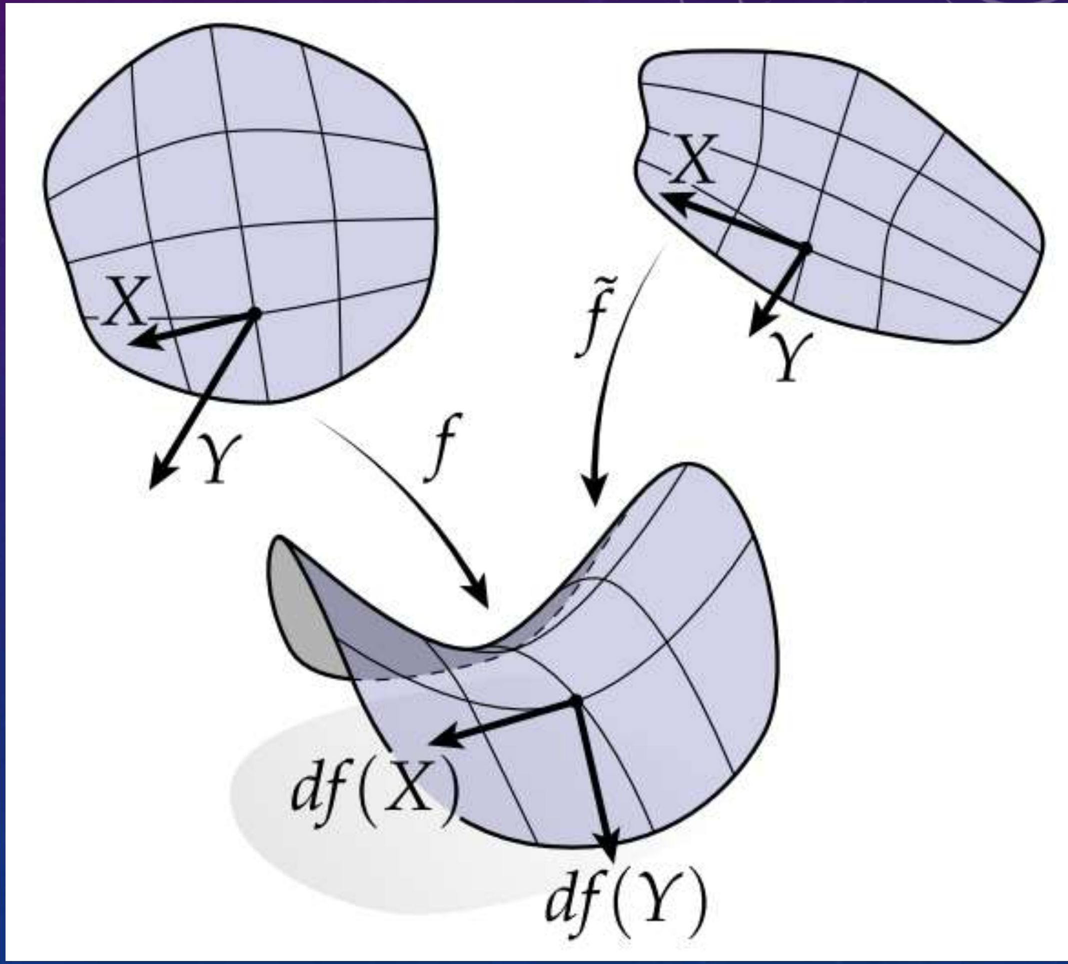
- Consider two parameterizations.
- Induce metric:

$$g(X,Y) = \langle df(X), df(Y)\rangle$$

$$= (J_f X)^T (J_f Y)$$

$$= X^T (J_f^T J_f) Y$$

First fundamental form $J_f^T J_f$



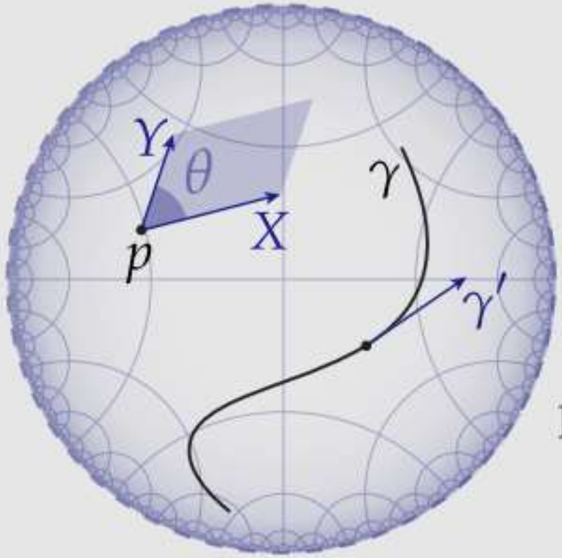
# Abstract Riemannian metric

- Induced Riemannian metric is just a (smoothly-varying) inner product at each point.
- We just write down some arbitrary smoothly-varying inner product.
- Inner product measures angles, lengths, areas, distances, …

**Example:** *hyperbolic metric* on unit disk.

$$U := \{p \in \mathbb{R}^2 : |p| < 1\}$$

$$g_p(X,Y) = \frac{4}{(1-|p|^2)^2}\langle X, Y\rangle$$




$$|X| = \sqrt{g_p(X,X)}$$

$$\theta = \arccos\left(g_p(X/|X|, Y/|Y|)\right)$$

$$\text{area}(X,Y) = \sqrt{\det(g_p)}(X \times Y)$$

$$\text{length}(\gamma) = \int_0^L g_{\gamma(s)}(\gamma', \gamma')^{1/2}\, ds$$

$$\ldots$$

# Embedding theorems

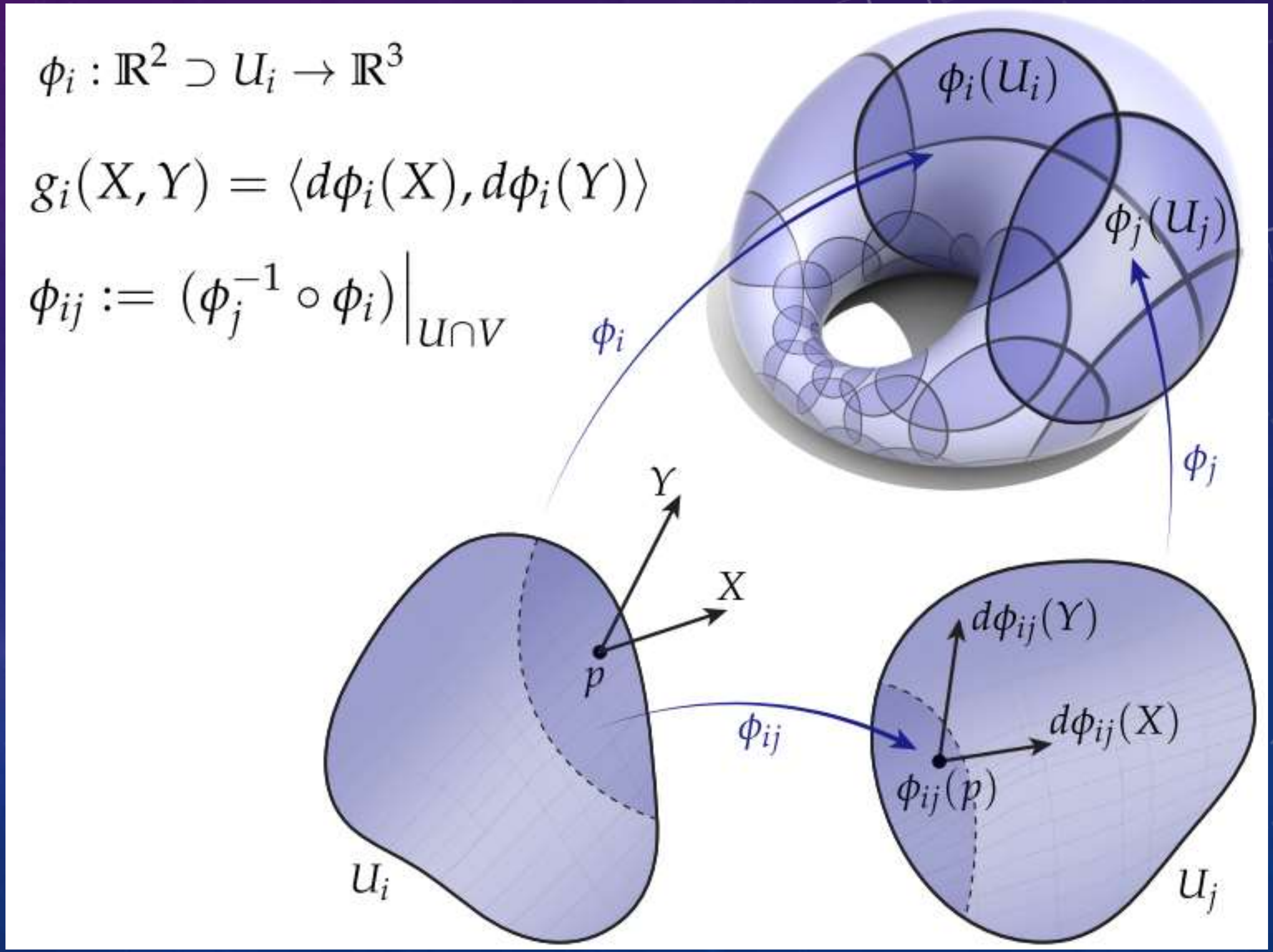
- Given a Riemannian metric g on region $\Omega$, can we find an embedding $f$ such that $g(X, Y) = \langle df(X), df(Y) \rangle$?
- Nash embedding theorems: always have global $C^k$ embedding in sufficiently high dimension.
- Most surfaces aren't easily expressed as the image of one parameterized "patch", e.g. how to find $\Omega$ for close surface?

# Atlas & charts

- Instead, cover a surface with overlapping patches ("charts").
- Each chart $\phi_i$ defines an induced Riemannian metric $g_i$ :

$$g_j\left(d\phi_{ij}(X), d\phi_{ij}(Y)\right)$$

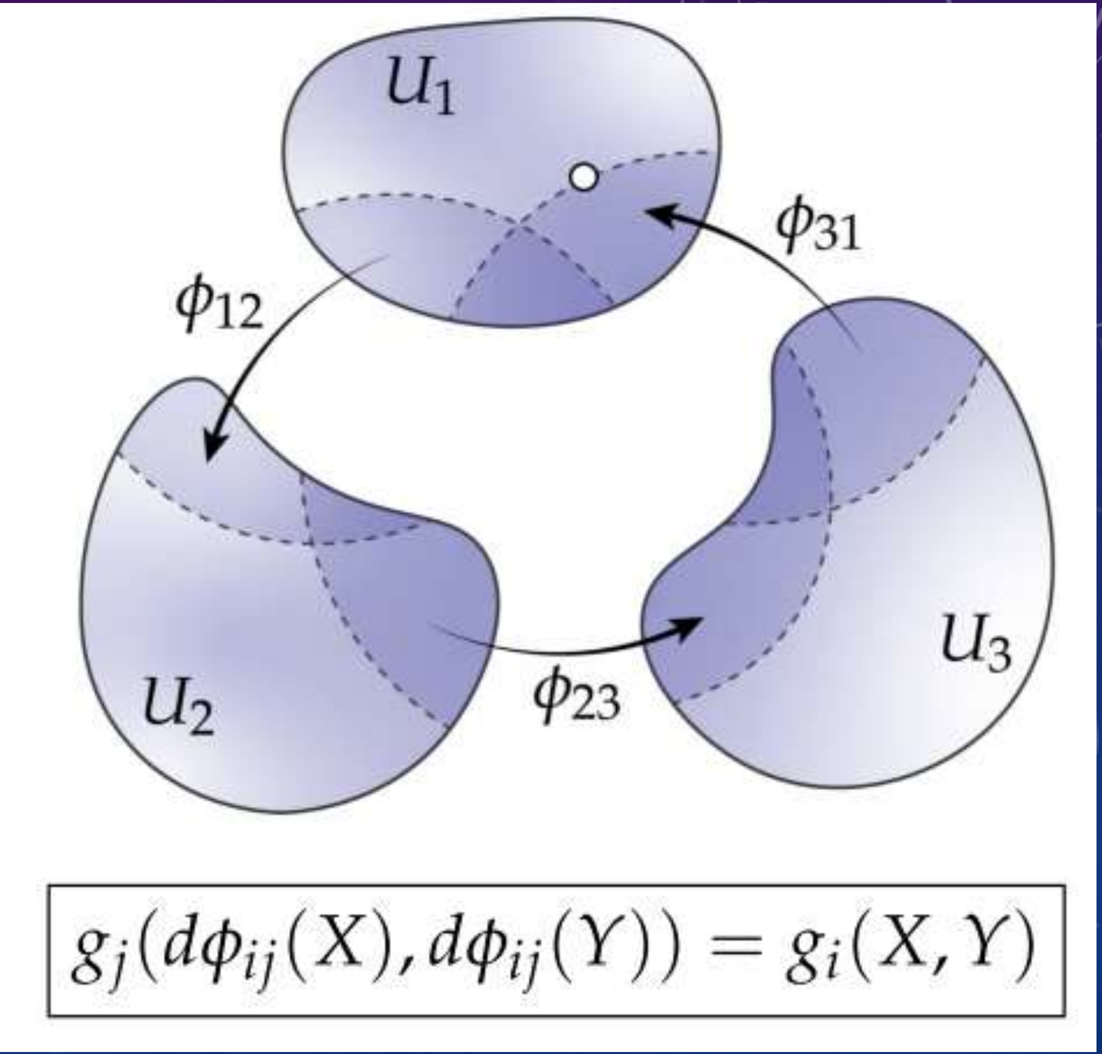
$$= g_i(X, Y)$$



# Riemann manifold

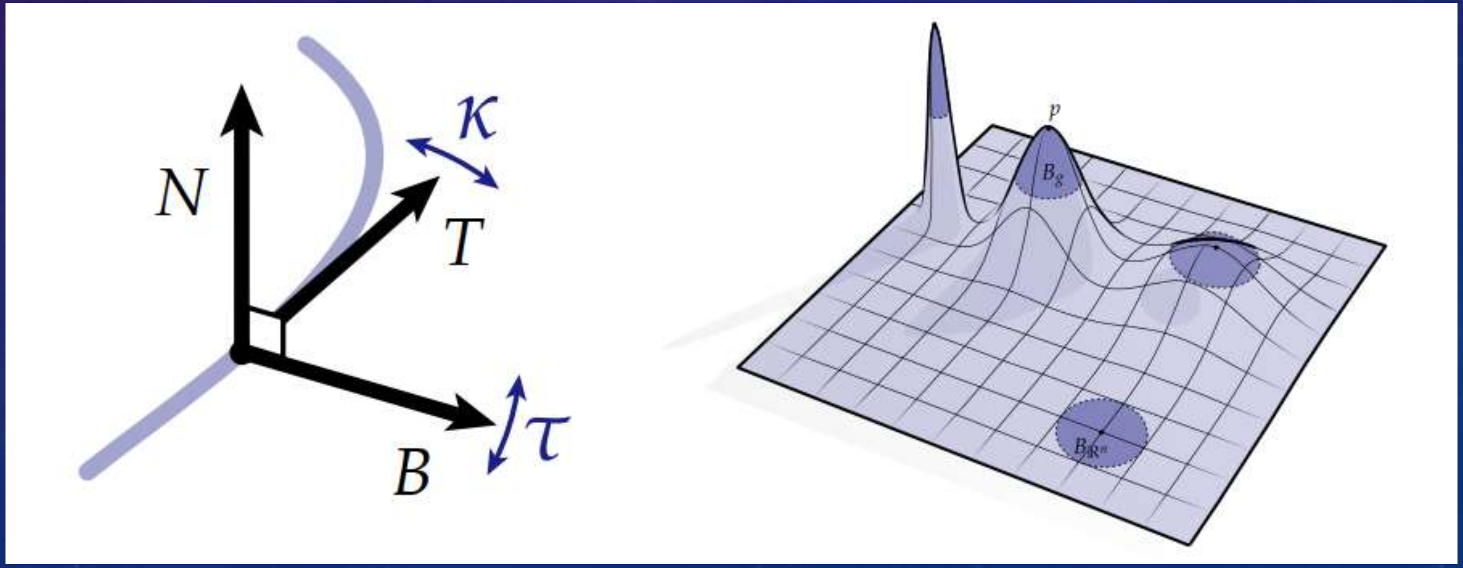
- Collection of open sets $\mathcal{U}_i \subset \mathbb{R}^2$
- Transition maps $\phi_{ij}$ on overlaps (differentiable both ways)
- local metric $g_i$ per patch, compatible on overlaps

Riemannian manifold $\mathcal{M}$ is “union” of all these pieces (do not need embeddings $\phi_i$)



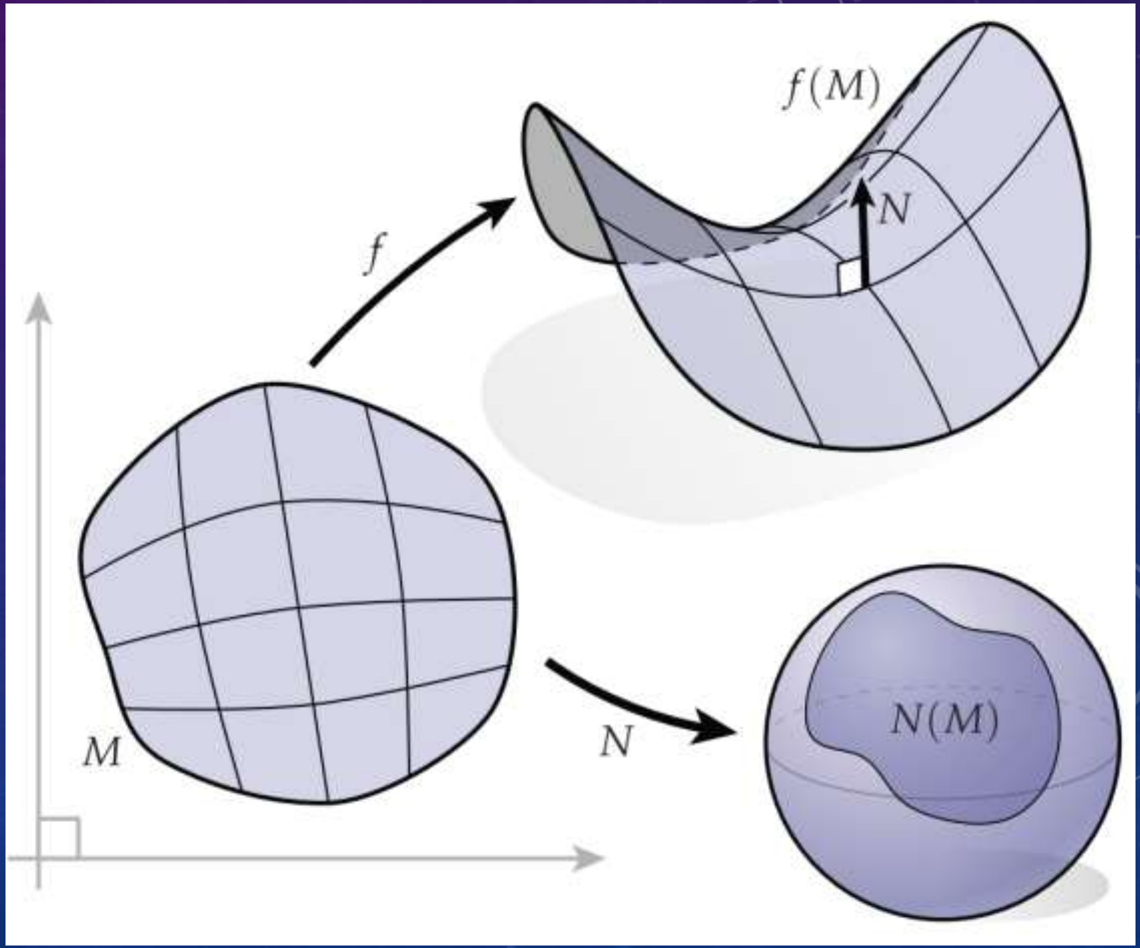
# Curvature

- Intuitively, describes "how much a shape bends"
  - Extrinsic: how quickly does the tangent plane/normal change?
  - Intrinsic: how much do quantities differ from flat case?



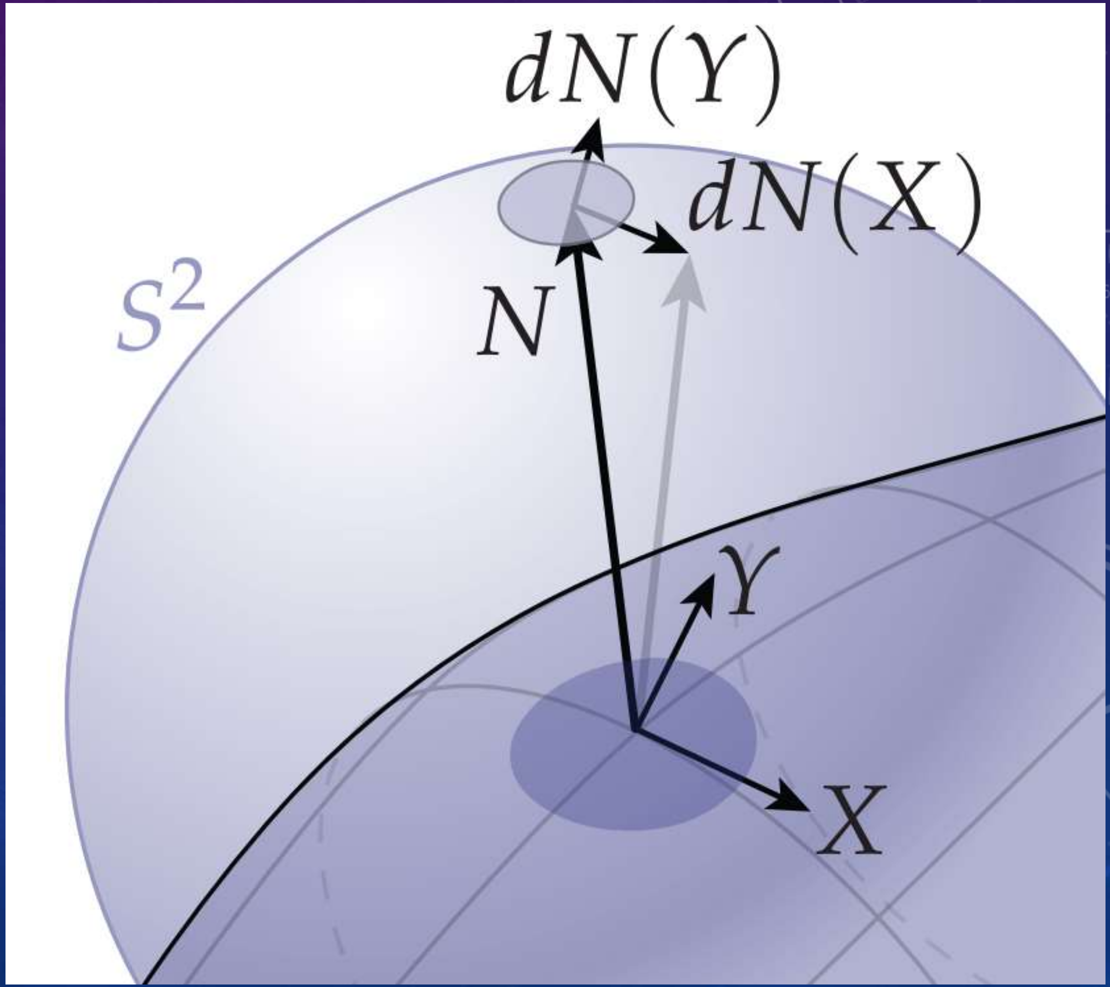
# Gauss map

- The Gauss map $N$ is a continuous map taking each point on the surface to a unit normal vector
- Visualize Gauss map as a map from the domain to the unit sphere



# Weingarten map

- The Weingarten map $dN$ is the differential of the Gauss map $N$
- At any point, $dN(X)$ gives the change in the normal vector along a given direction $X$
- $\langle dN(X), N \rangle = 0$, for any $X$



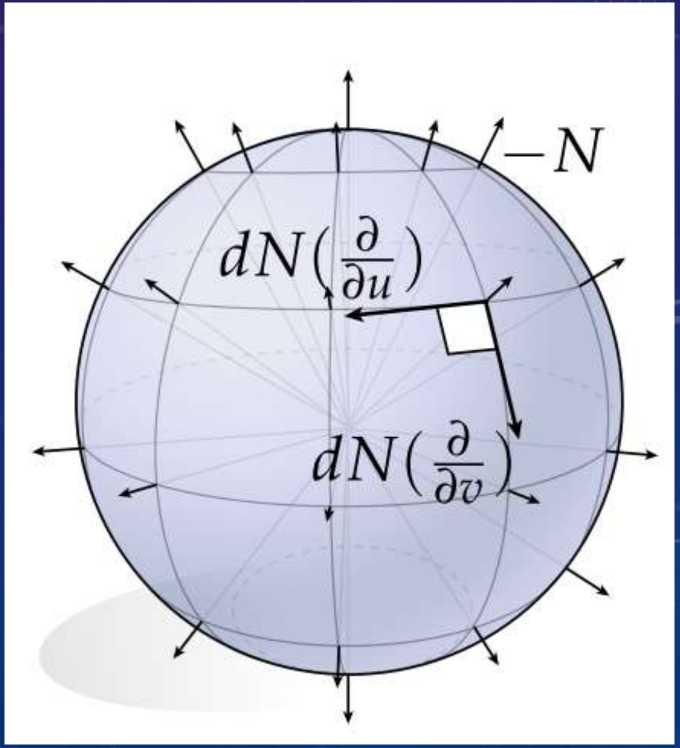
# Weingarten map—example

Recall that for the sphere, $N = -f$. Hence, Weingarten map $dN$ is just $-df$

$$f = (\cos u \sin v, \sin u \sin v, \cos v)$$

$$df = (-\sin u \sin v, \cos u \sin v, \cos v)du$$
$$+(\cos u \cos v, \sin u \cos v, -\sin v)dv$$

$$dN = (\sin u \sin v, -\cos u \sin v, -\cos v)du$$
$$+(-\cos u \cos v, -\sin u \cos v, \sin v)dv$$



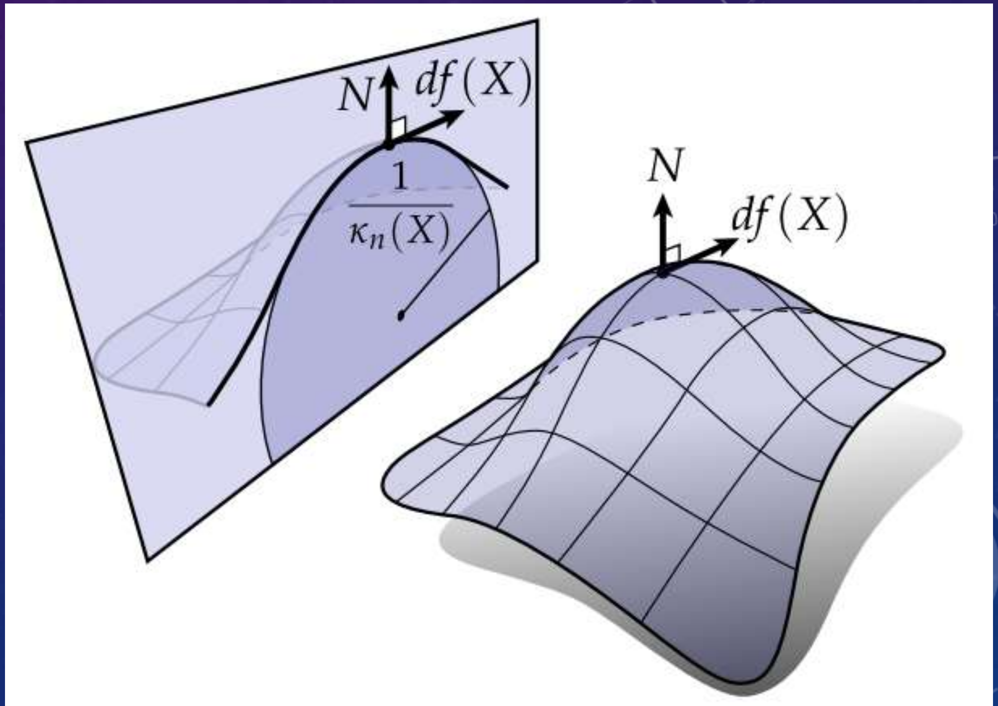
# Normal curvature

Curves: rate of change of the tangent.

Surfaces: how quickly the normal is changing.

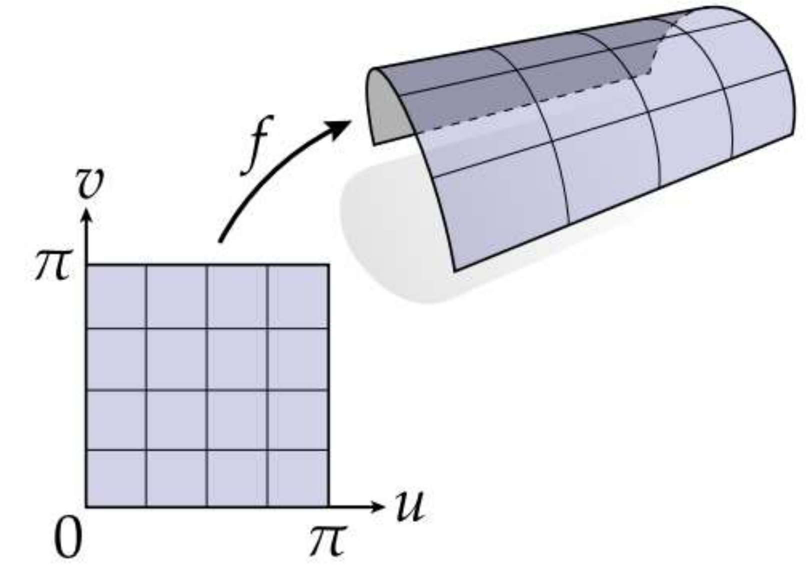
Normal curvature is rate at which normal is bending along a given tangent direction:

$$\kappa_N(X) = \frac{\langle df(X), dN(X) \rangle}{|df(X)|\text{^}2}$$




Equivalent to intersecting surface with normal-tangent plane and measuring the usual curvature of a plane curve.

# Normal curvature—example




Consider a parameterized cylinder:

$$f(u,v) = (\cos u, \sin u, v)$$

$$df = (-\sin u, \cos u, 0)du + (0,0,1)dv$$

$$N = (-\sin u, \cos u, 0) \times (0,0,1) = (\cos u, \sin u, 0)$$

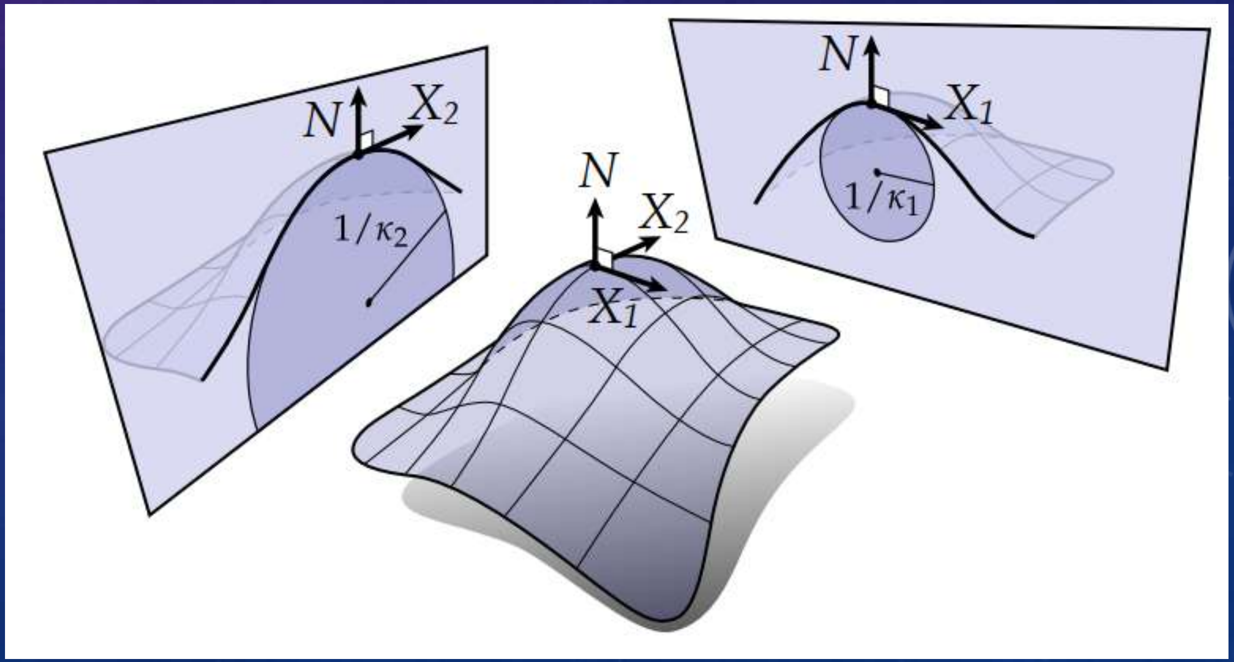
$$dN = (-\sin u, \cos u, 0)du$$

$$\kappa_N\left(\frac{\partial}{\partial u}\right) = \frac{\left\langle df\left(\frac{\partial}{\partial u}\right), dN\left(\frac{\partial}{\partial u}\right)\right\rangle}{\left|df\left(\frac{\partial}{\partial u}\right)\right|^2} = \frac{\langle(-\sin u, \cos u, 0), (-\sin u, \cos u, 0)\rangle}{|(-\sin u, \cos u, 0)|^2} = 1,\ \kappa_N\left(\frac{\partial}{\partial v}\right) = 0$$

# Principal curvature

Among all directions $X$, there are two principal directions $X_1, X_2$ where normal curvature has minimum/maximum value (respectively).

1. $\mathrm{g}(X_1, X_2) = 0$
2. $dN(X_i) = \kappa_i df(X_i)$



# Shape operator

The change in the normal $N$ is always tangent to the surface: $\langle dN, N \rangle = 0$.

Therefore must be some linear map $S$ from tangent vectors to tangent vectors, called the shape operator, such that $df(SX) = dN(X)$.

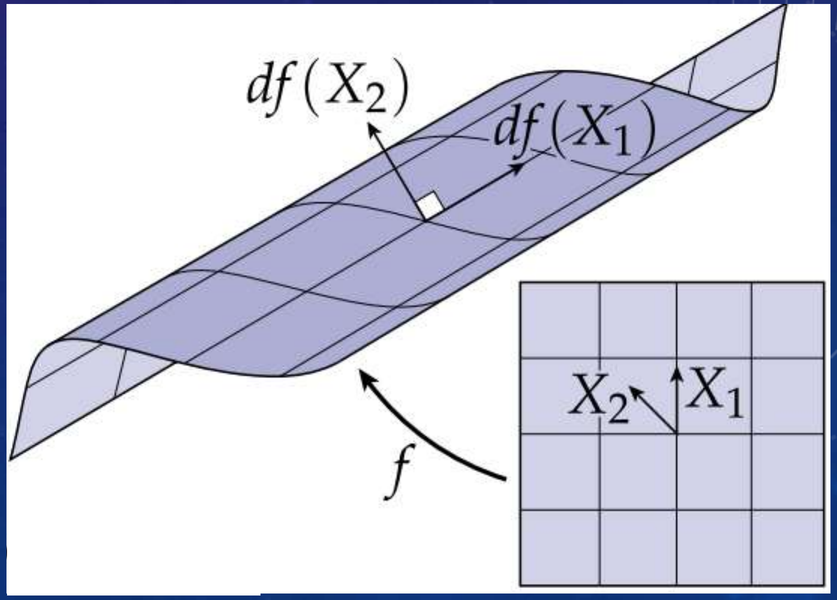
- Principal directions are the eigenvectors of $S$.
- Principal curvatures are eigenvalues of $S$.

Note: $S$ is not a symmetric matrix! Hence, eigenvectors are not orthogonal in $\mathbb{R}^2$; only orthogonal with respect to induced metric g.

# Shape operator—example

Consider a nonstandard parameterized cylinder: $f(u,v) = (\cos u, \sin u, u+v)$ $df = (-\sin u, \cos u, 1)du + (0,0,1)dv$ $N = (\cos u, \sin u, 0)$ $dN = (-\sin u, \cos u, 0)du$

$$df(SX) = dN(X) \qquad \begin{bmatrix} -\sin u & 0 \\ \cos u & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} -\sin u & 0 \\ \cos u & 0 \\ 0 & 0 \end{bmatrix}$$




$$S = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \qquad [X_1, X_2] = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

$$df(X_1) = (0,0,1) \qquad \kappa_1 = 0$$
$$df(X_2) = (\sin u, -\cos u, 0) \qquad \kappa_2 = 1$$

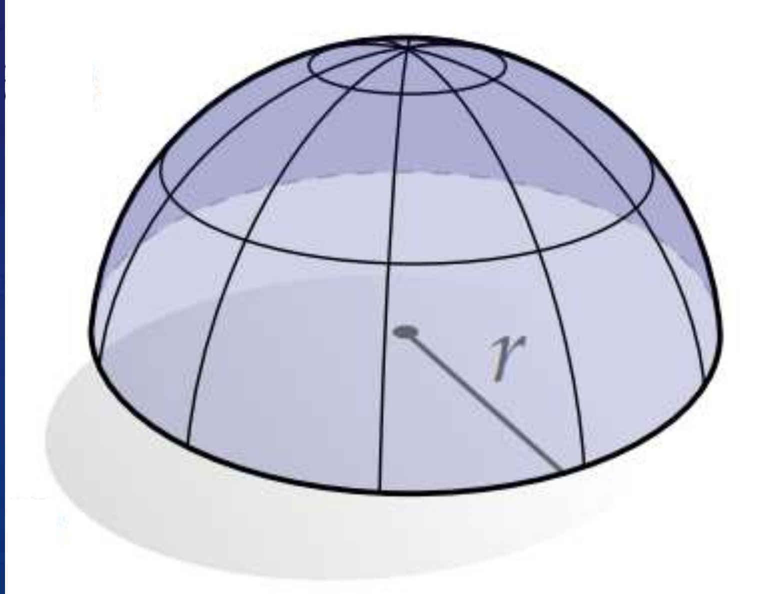
# Umbilic points

Points where principal curvatures are equal are called umbilic points

Principal directions are not uniquely determined here

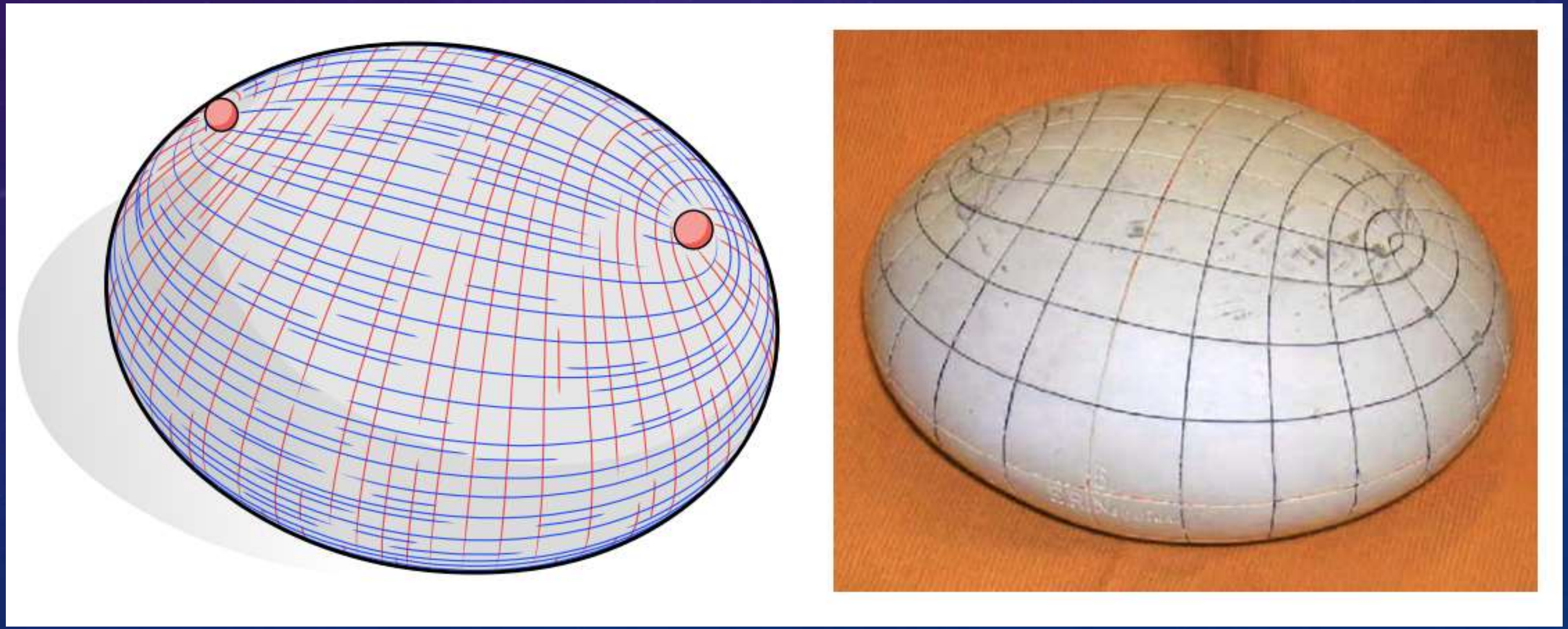
$$S = \begin{bmatrix} 1/r & 0 \\ 0 & 1/r \end{bmatrix} \qquad \kappa_1 = \kappa_2 = \frac{1}{r}$$

$$\forall X, SX = \frac{1}{r} X$$



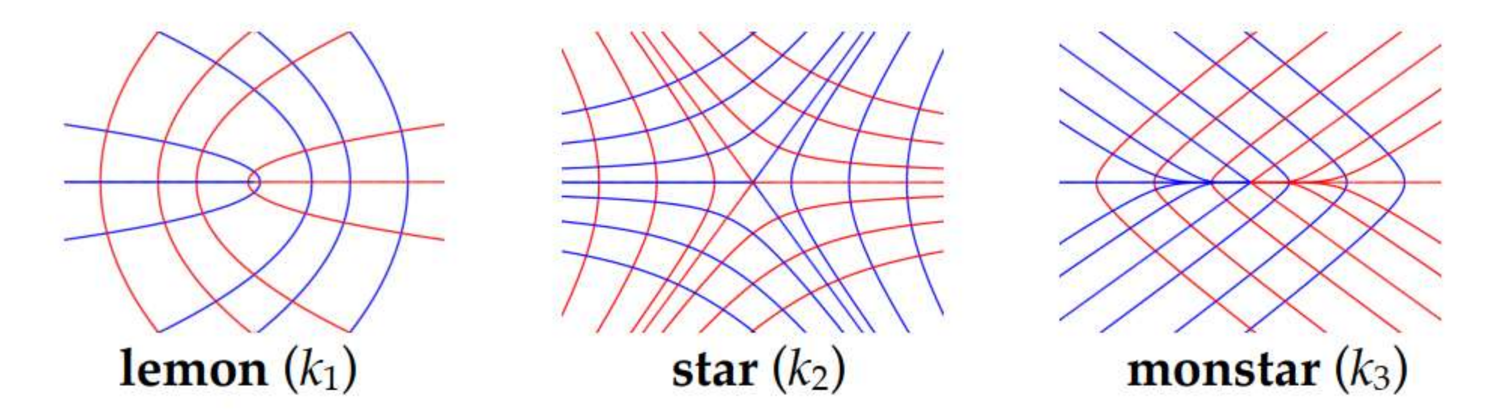
# Principal curvature nets

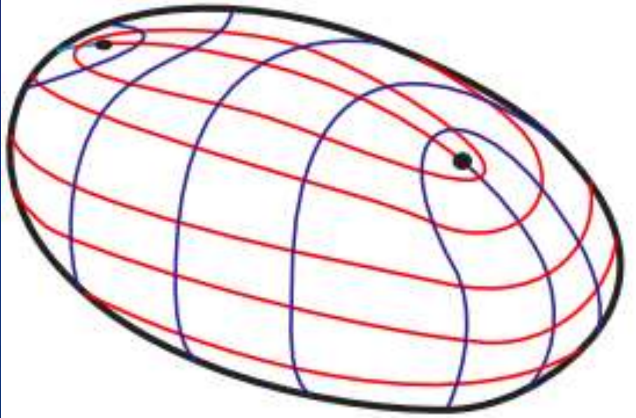
- Walking along principal direction field yields principal curvature lines.
- Collection of all such lines is called the principal curvature network



# Topological invariance of Umbilic count

- Classify regions around (isolated) umbilic points into three types based on behavior of principal network.
- If $k_1, k_2, k_3$ are number of umbilics of each type, then $k_1 - k_2 + k_3 = 2\chi$

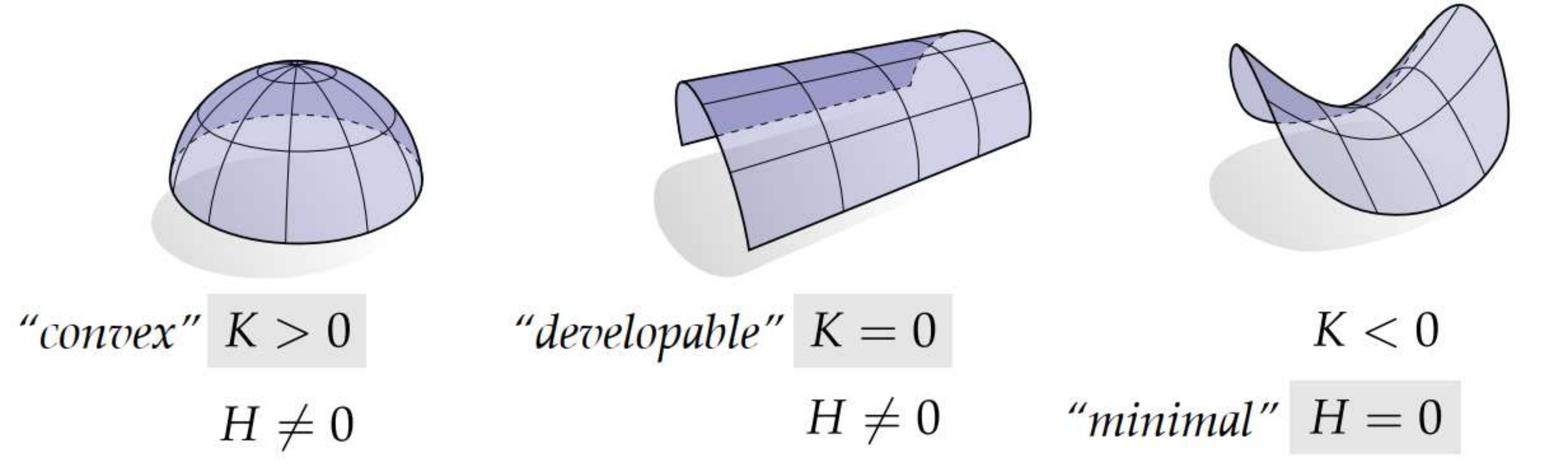


# Gaussian and mean curvature

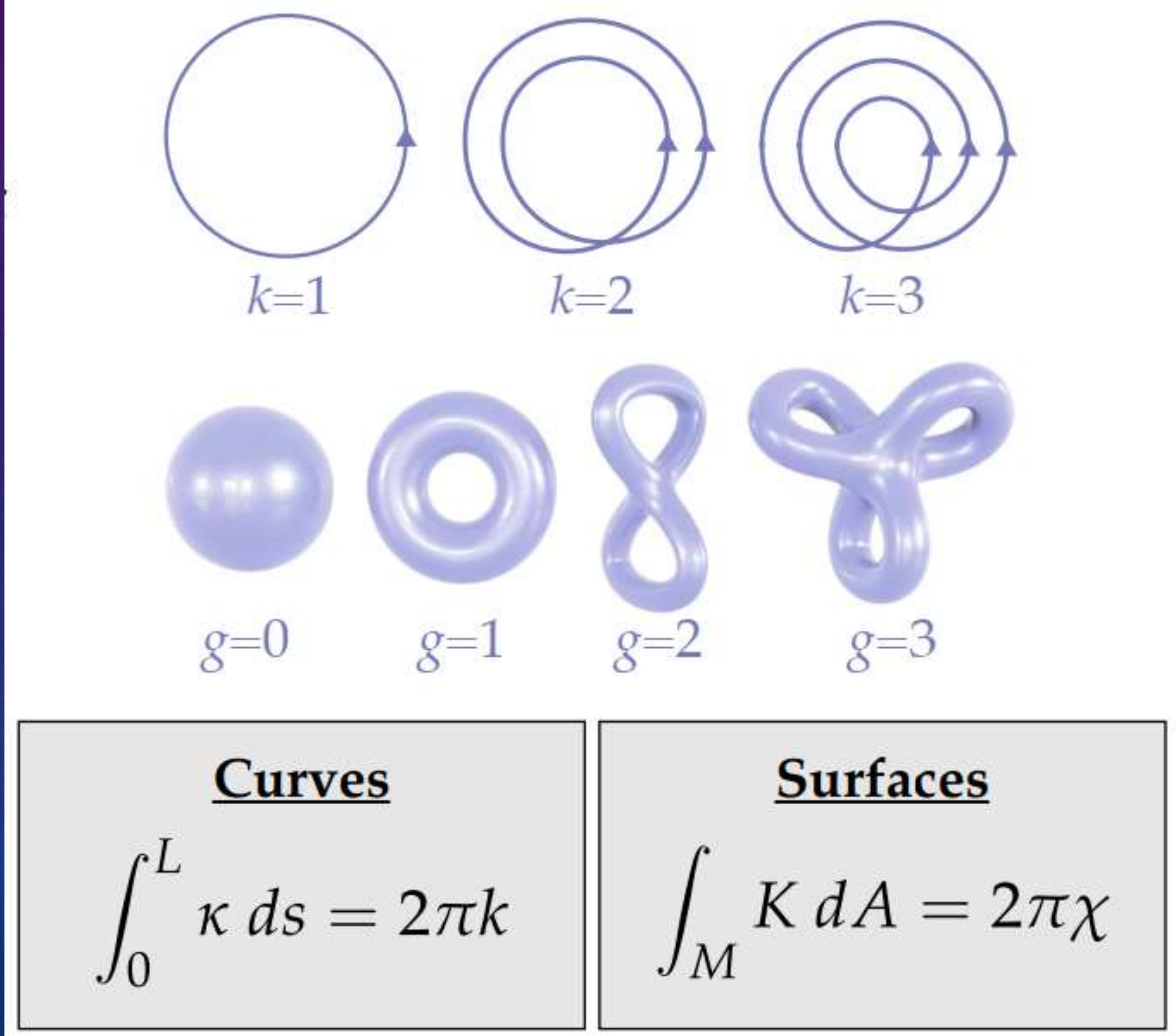
- Gaussian and mean curvature also fully describe local bending

Gaussian curvature: $K = \kappa_1 \kappa_2$ Mean curvature: $H = \frac{1}{2}(\kappa_1 + \kappa_2)$



# Gauss-Bonnet theorem

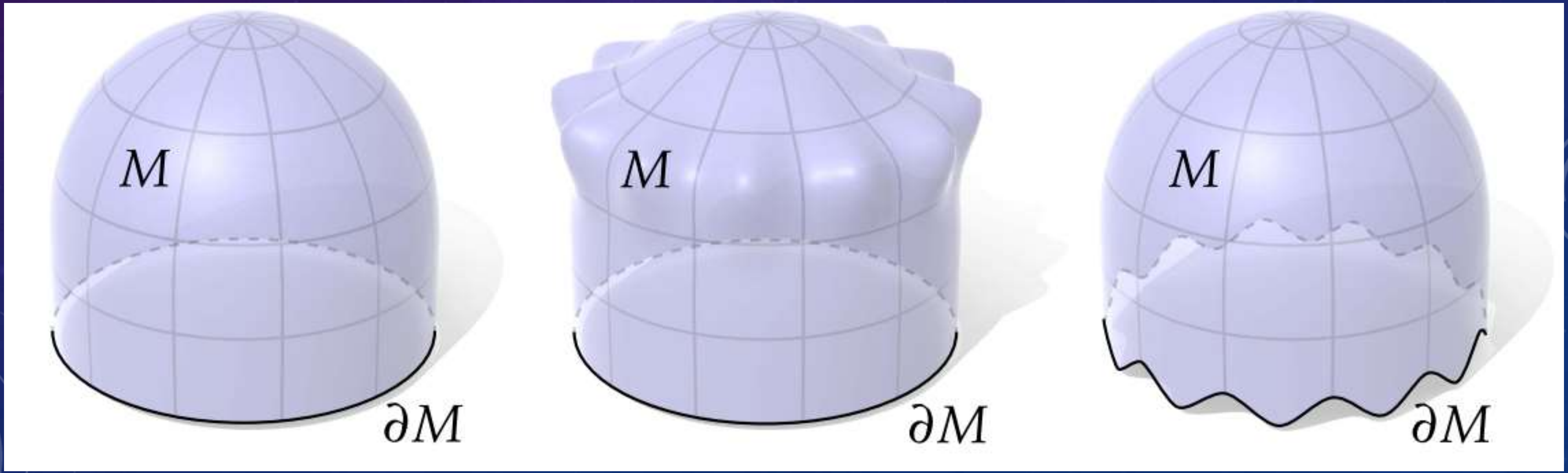
- Recall that the total curvature of a closed plane curve was always equal to 2π times turning number $k$.
- For surfaces, Gauss-Bonnet theorem says total Gaussian curvature is always 2π times Euler characteristic $\chi = 2 - 2g$



# Gauss-Bonnet theorem with boundary

- Generalize to surfaces with boundary:

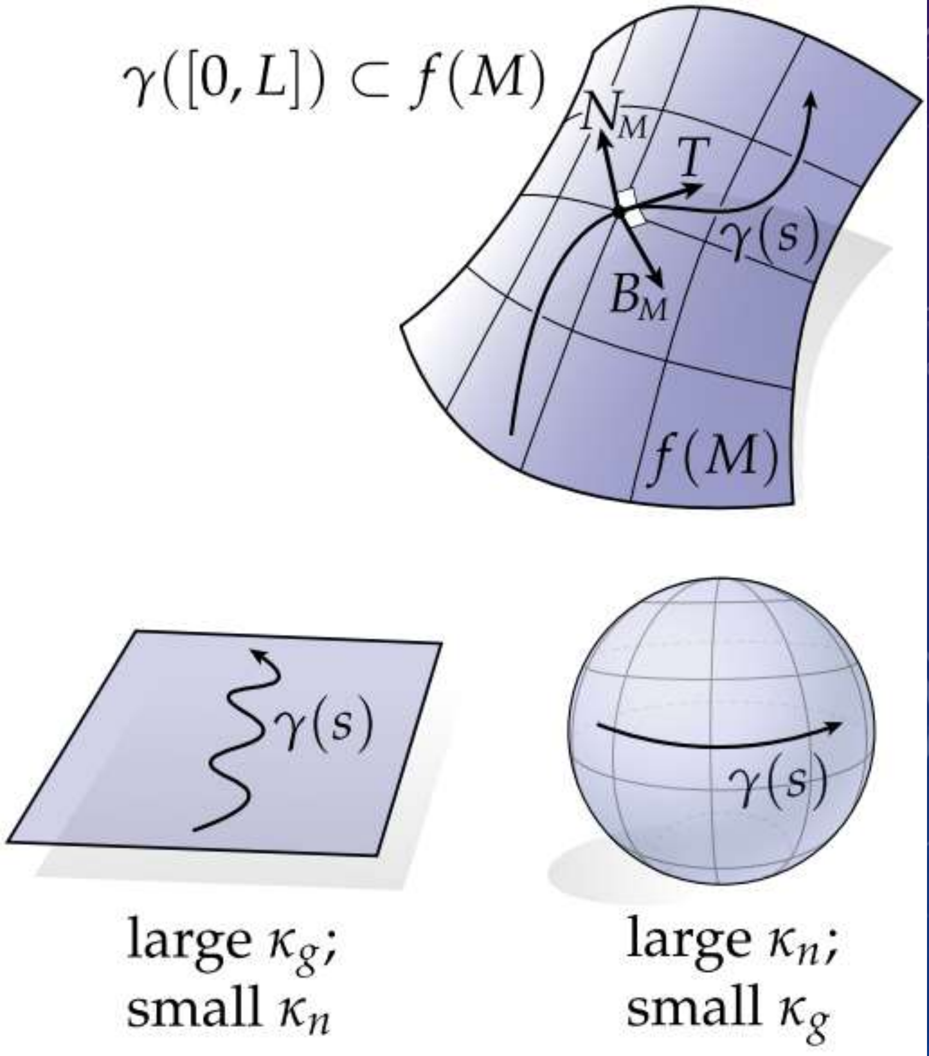
$$\int_M K dA + \int_{\partial M} \kappa_g ds = 2\pi\chi, \; \chi = 2 - 2g - b$$



# Curvature of a curve in a surface

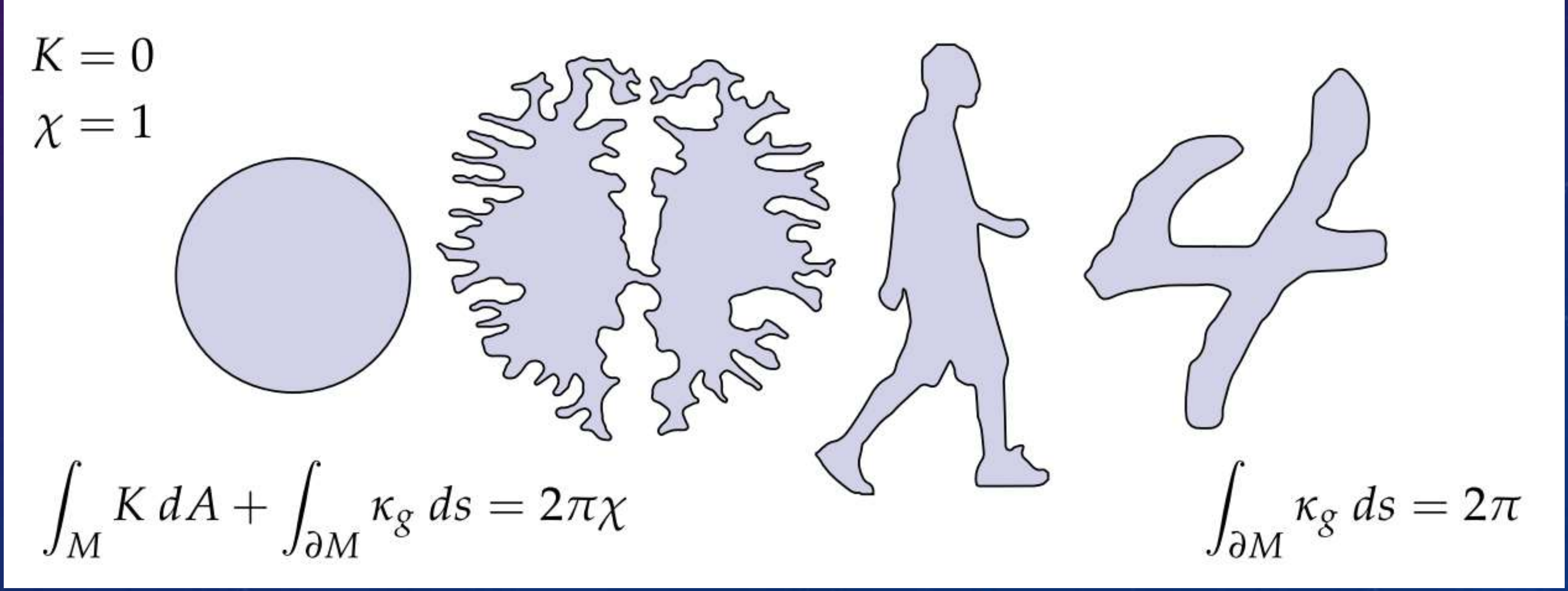
- Broke the "bending" of a space curve into curvature $\kappa$ and torsion $\tau$
- For a curve in a surface, can instead break into normal and geodesic curvature

$$\kappa_n = \left\langle N_M, \frac{dT}{ds} \right\rangle, \qquad \kappa_g = \left\langle B_M, \frac{dT}{ds} \right\rangle$$



# Example: planar disk

- For a disk in the plane, total curvature of boundary is equal to 2π (turning number theorem)



# Mean curvature

- Lemma. Normal curvature along $Y = \cos\theta\, Y_1 + \sin\theta\, Y_2$, $Y_1, Y_2$ principal directions,

$$\kappa_N|_Y = \cos^2\theta\, \kappa_1 + \sin^2\theta\, \kappa_2$$

$$df(SX) = dN(X).$$

- Theorem. The mean curvature is the normal curvature averaged over all directions $Y = \cos\theta\, X_1 + \sin\theta\, X_2$, where $X_1, X_2$ are an orthonormal basis of tangent plane,

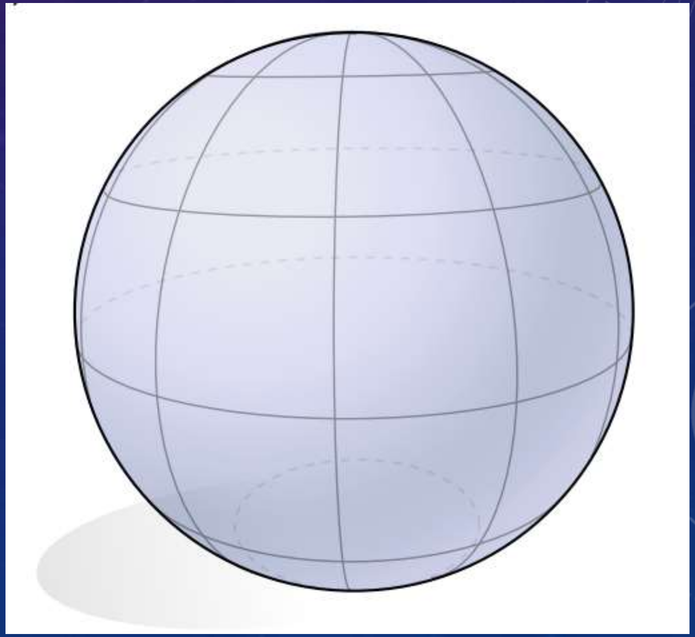
$$H = \frac{1}{2\pi}\int_0^{2\pi} \kappa_N\big(Y(\theta)\big)d\theta$$

# Total mean curvature?

- Theorem. (Minkowski): for a convex surface,

$$\int_M H \, dA \geq \sqrt{4\pi A}$$

When the shape is a sphere, equality satisfies.



# First and second fundamental form

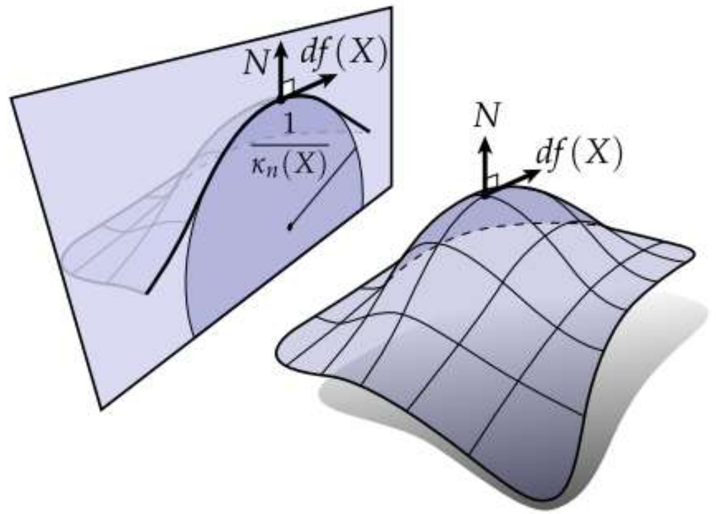
- Fundamental Theorem:

Two surfaces in $\mathbb{R}^3$ are identical up to rigid motions if and only if they have the same first and second fundamental forms

Not every pair of bilinear forms $I, II$ describes a valid surface—must satisfy the Gauss Codazzi equations

$$\mathbf{I}(X,Y) := \langle df(X), df(Y) \rangle$$

$$\mathbf{II}(X,Y) := \langle dN(X), df(Y) \rangle$$




$$\kappa_N(X) = \frac{\langle df(X), dN(X) \rangle}{|df(X)|^2} = \frac{\mathbf{II}(X,X)}{\mathbf{I}(X,X)}$$

# Descriptions of Surfaces

- What data is sufficient to completely determine a surface in space?
  - First & second fundamental form (Gauss-Codazzi)
  - Mean curvature and metric (up to "Bonnet pairs")
  - Convex surfaces: metric alone is enough (Alexandrov/Pogorolev)
  - Gauss curvature essentially determines metric (Kazdan-Warner)
  - …

# Discrete surface

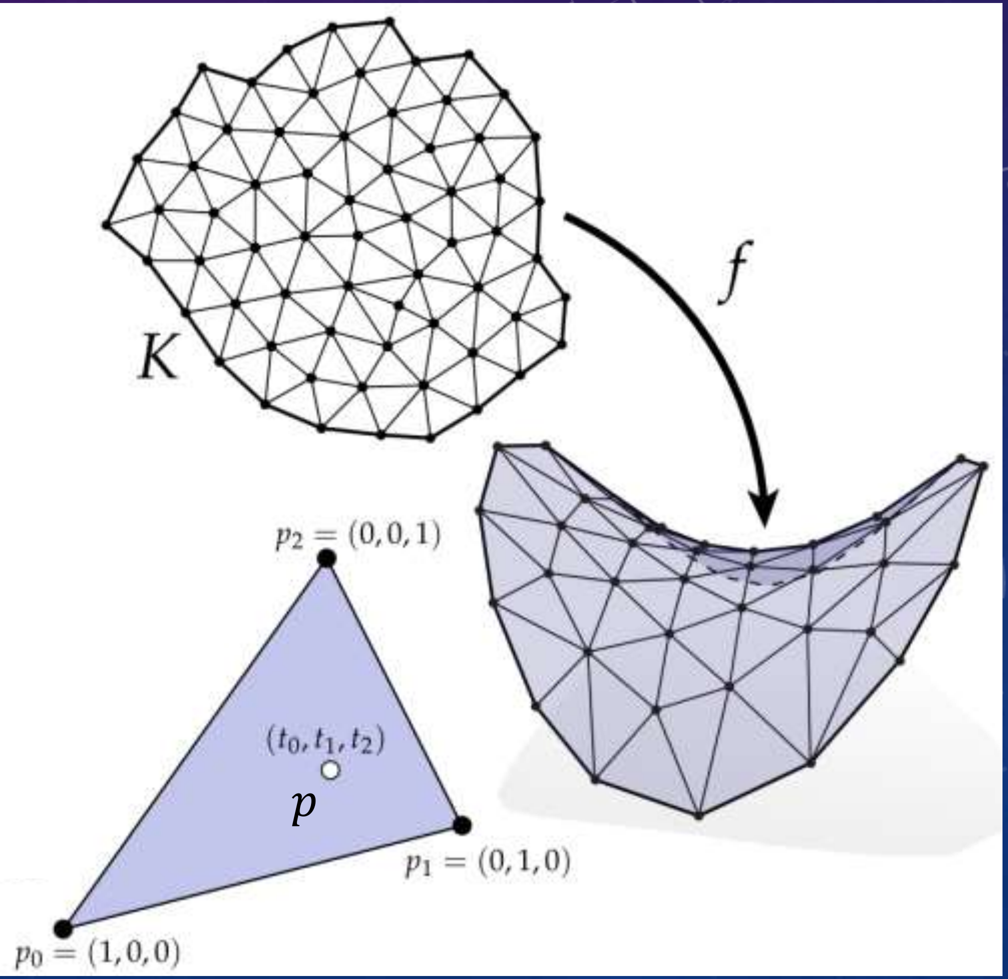
# Piecewise linear approximation

Coordinate $f_i$ of each vertex

Linear interpolate via barycentric coordinate

$$\begin{cases} t_0 = s_{\Delta p p_1 p_2} / s_{\Delta p_0 p_1 p_2} \\ t_1 = s_{\Delta p p_2 p_0} / s_{\Delta p_0 p_1 p_2} \\ t_2 = s_{\Delta p p_0 p_1} / s_{\Delta p_0 p_1 p_2} \end{cases}$$

$$f(p) = t_0 f_0 + t_1 f_1 + t_2 f_2,$$
$$t_0 + t_1 + t_2 = 1$$

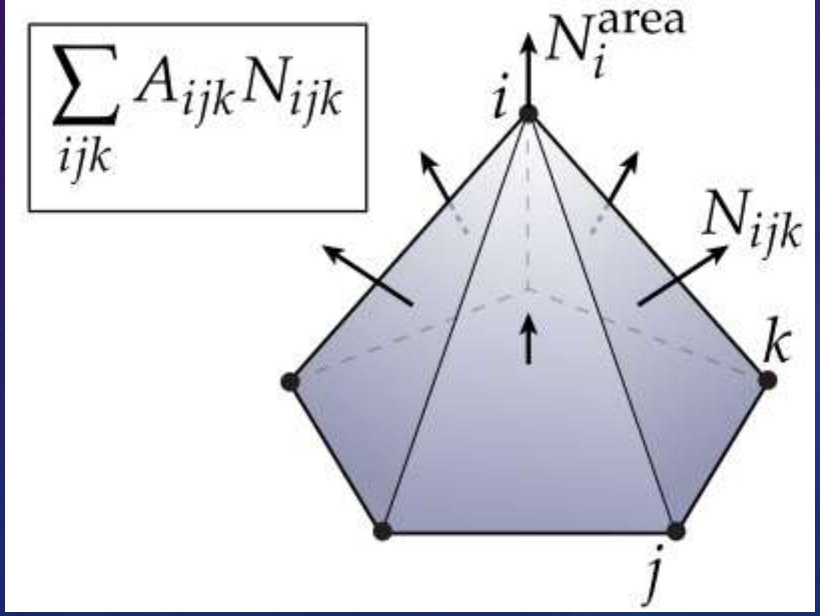


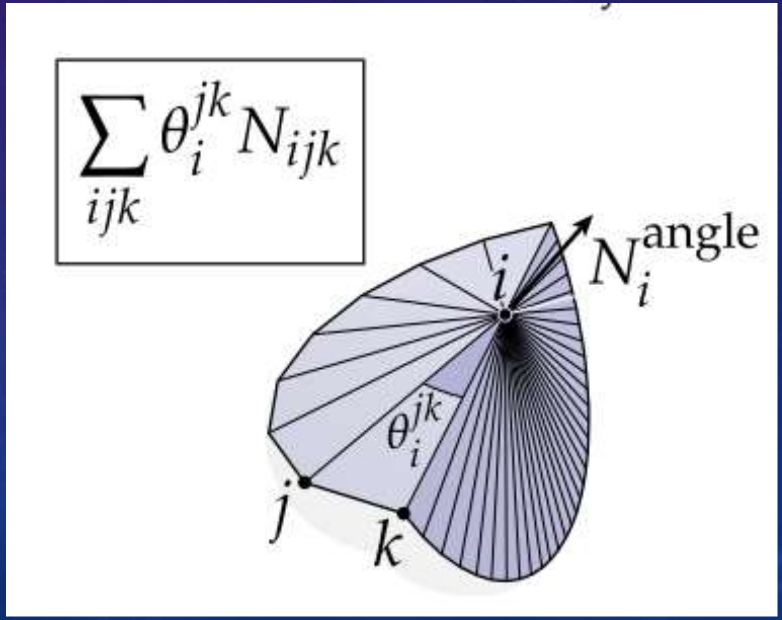
# Discretization

- Differential → edge vector: $(df)_{ij} = f_j - f_i$
- Discrete tangent: $T_{ijk} = \{(df)_{ij}, (df)_{jk}\}$.
- Discrete face normal : $N_{ijk} = \frac{(df)_{ij} \times (df)_{jk}}{|(df)_{ij} \times (df)_{jk}|}$

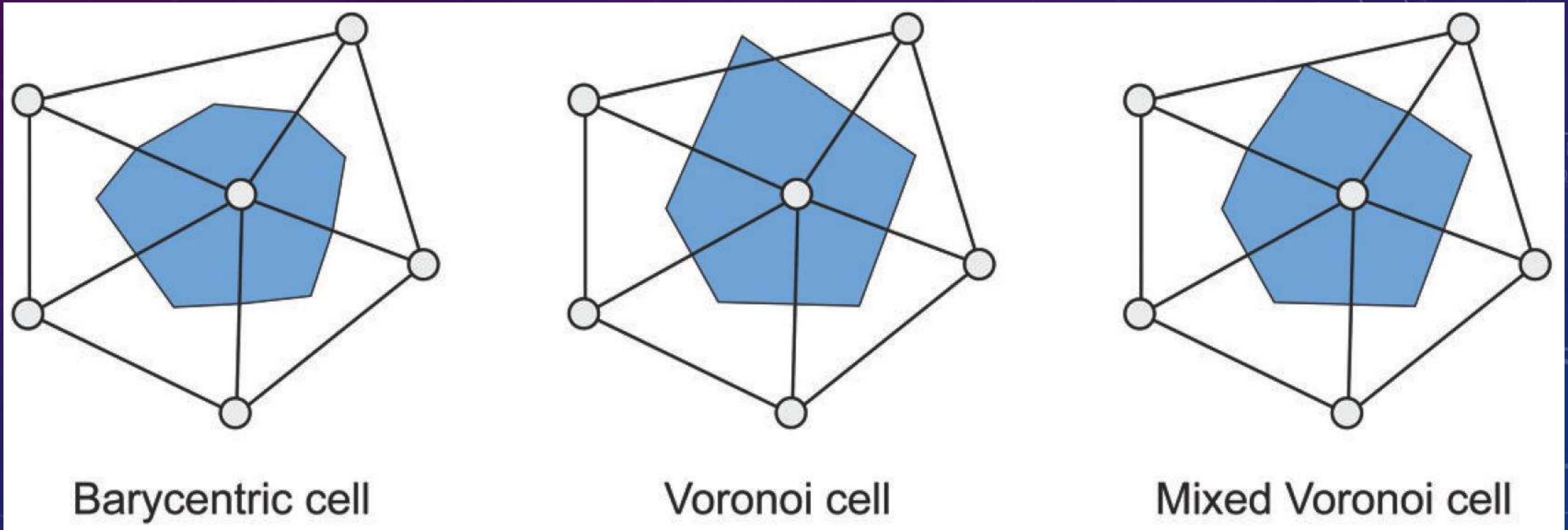
# Vertex normal

- Area weighted vertex normal and angle weighted vertex normal



# Local averaging region




triangle barycenters edge midpoints

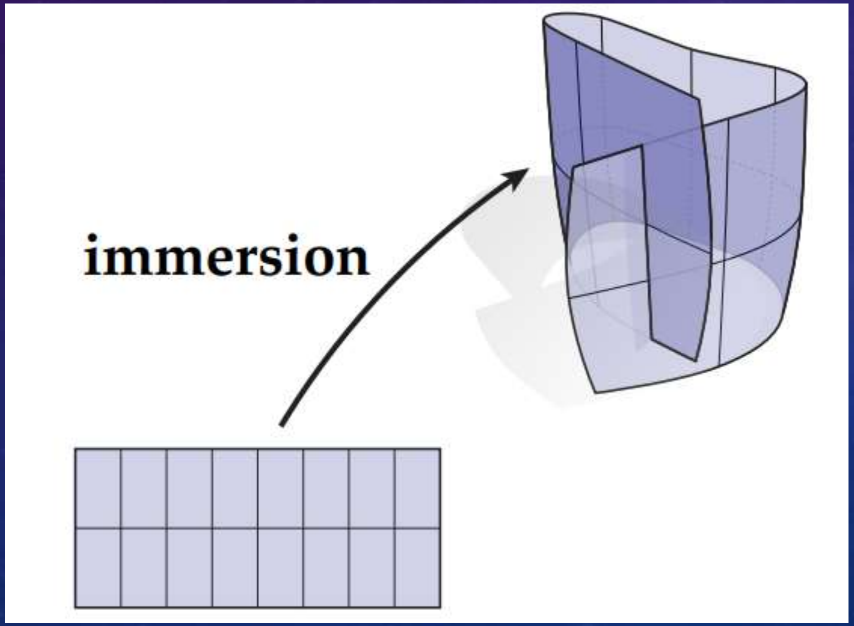
triangle barycenters → triangle circumcenter

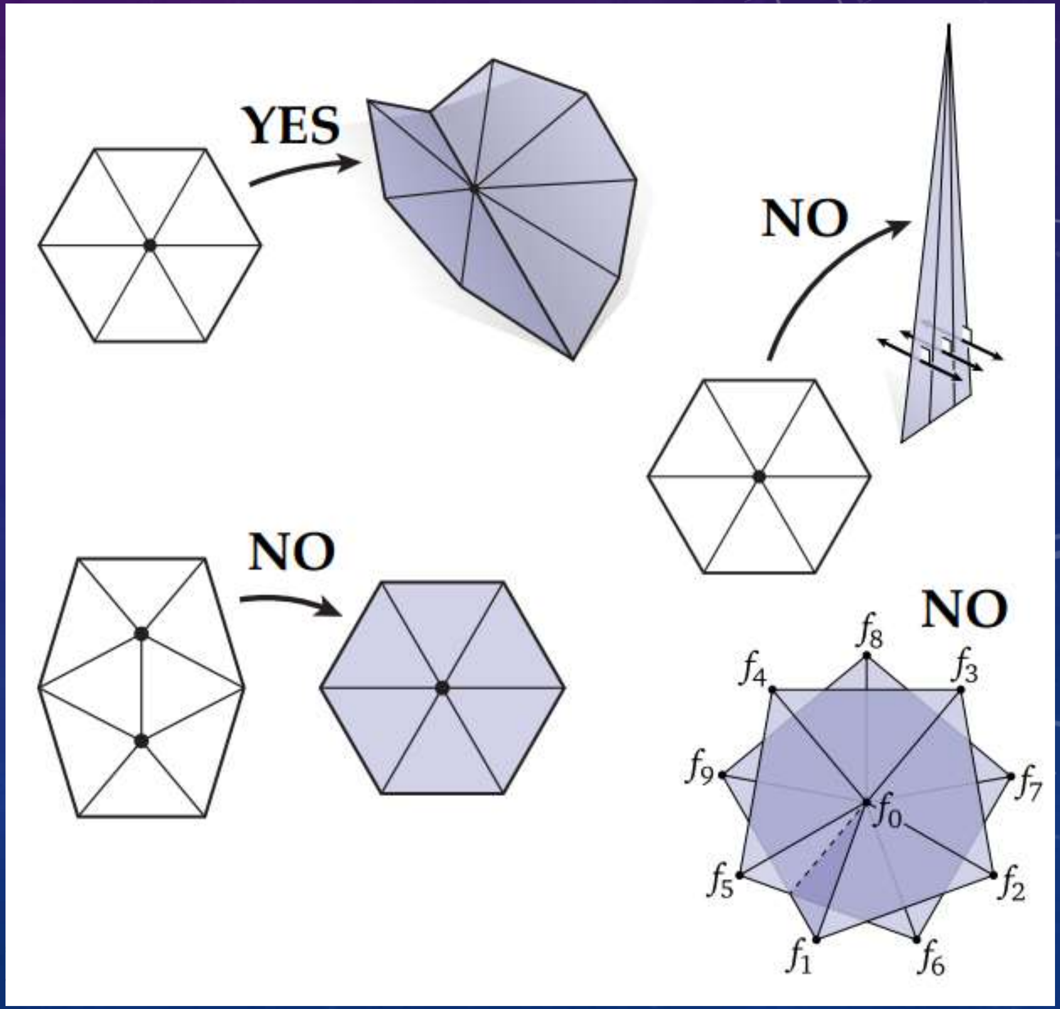
circumcenter for obtuse triangles → edge midpoints

# Discrete regular (immersion)

- Local injectivity: $\left|J_f\right| \neq 0 \Leftrightarrow$

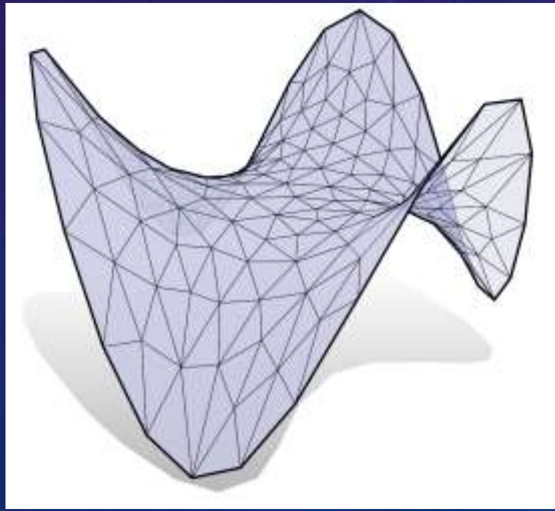
  $df(X) = 0$ if and only if $X = 0$



# Discrete Riemann metric

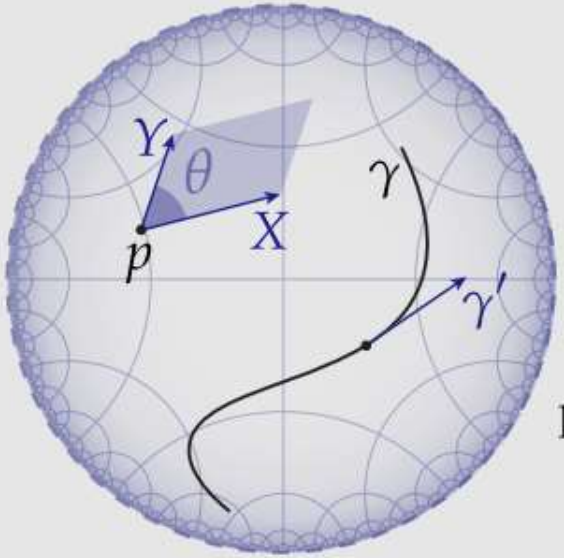
- Inner product measures angles, lengths, areas, distances, …
- For triangular mesh: $\{l_{ij}\}$



**Example:** *hyperbolic metric* on unit disk.

$$U := \{p \in \mathbb{R}^2 : |p| < 1\}$$

$$g_p(X,Y) = \frac{4}{(1-|p|^2)^2}\langle X,Y\rangle$$




$$|X| = \sqrt{g_p(X,X)}$$

$$\theta = \arccos\left(g_p(X/|X|, Y/|Y|)\right)$$

$$\text{area}(X,Y) = \sqrt{\det(g_p)}(X \times Y)$$

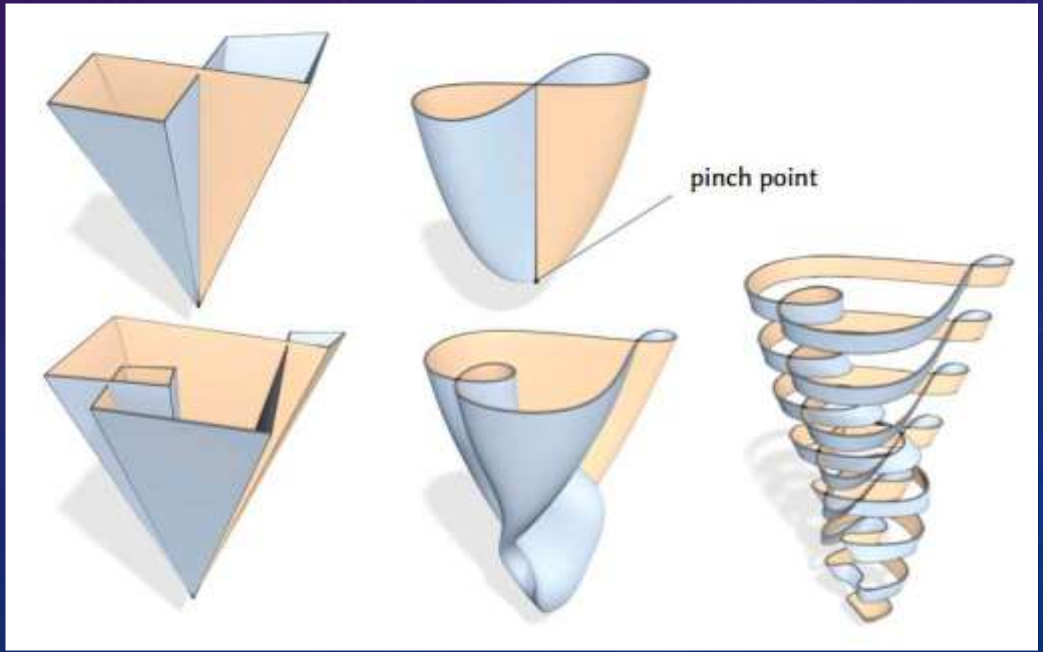
$$\text{length}(\gamma) = \int_0^L g_{\gamma(s)}(\gamma',\gamma')^{1/2}\, ds$$

…

# Recovery from metric

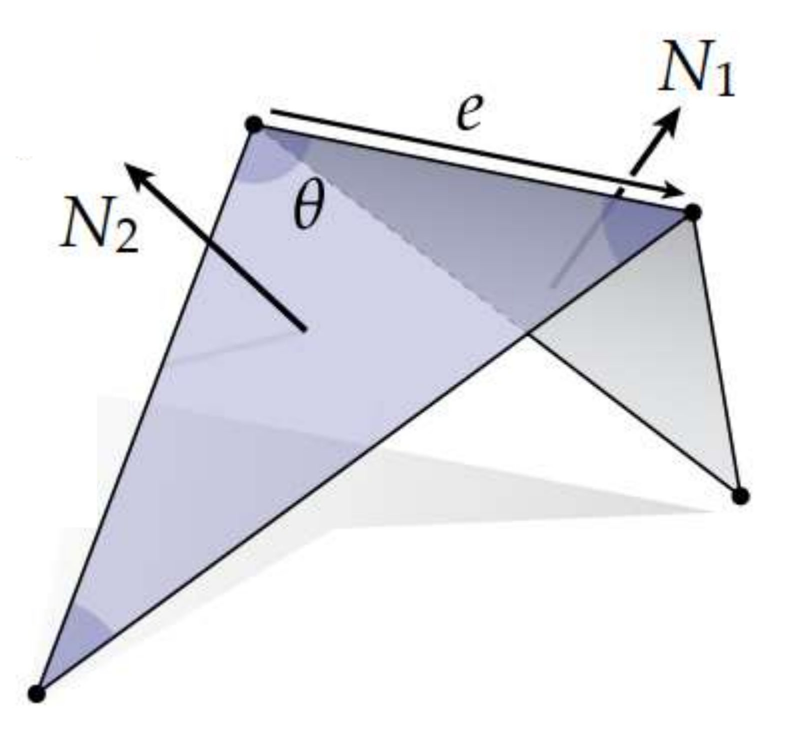
- Recovers mesh from lengths : Chern et al, “Shape from Metric” (2018)

Get deeper into discrete surfaces: discrete immersion, discrete spin structure …

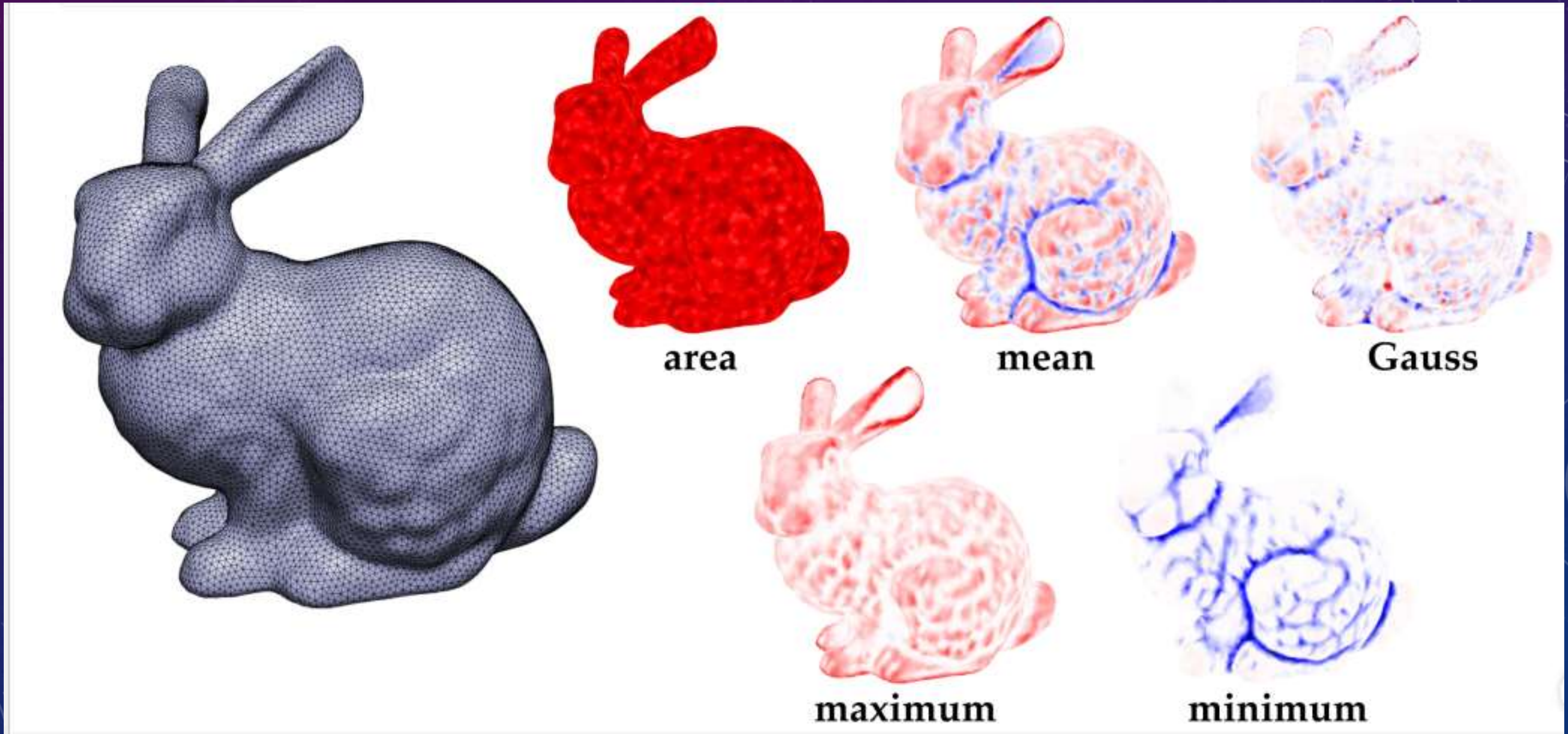



# Recovery from face normals

- Cross product of normals gives edge directions
- Dot product of edges gives interior angles
- Three angles determine triangle up to scale; normal determines plane of each triangle
- Build triangles one-by-one and "glue" together



# Discrete Curvature

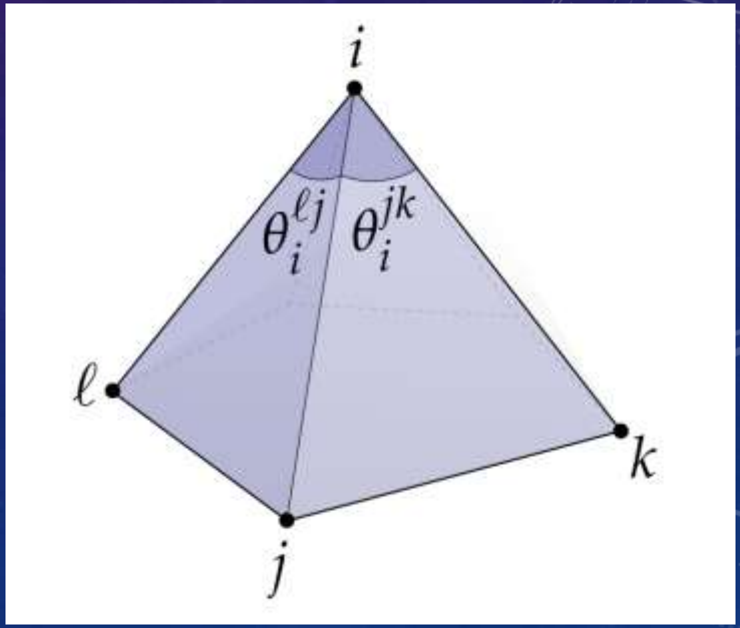


# Angle Defect

- The angle defect at a vertex $i$ is the deviation of the sum of interior angles from the Euclidean angle sum of $2\pi$:

$$\Omega_i = 2\pi - \sum_{ijk} \theta_i^{jk}$$

Measure how "flat" is the vertex.

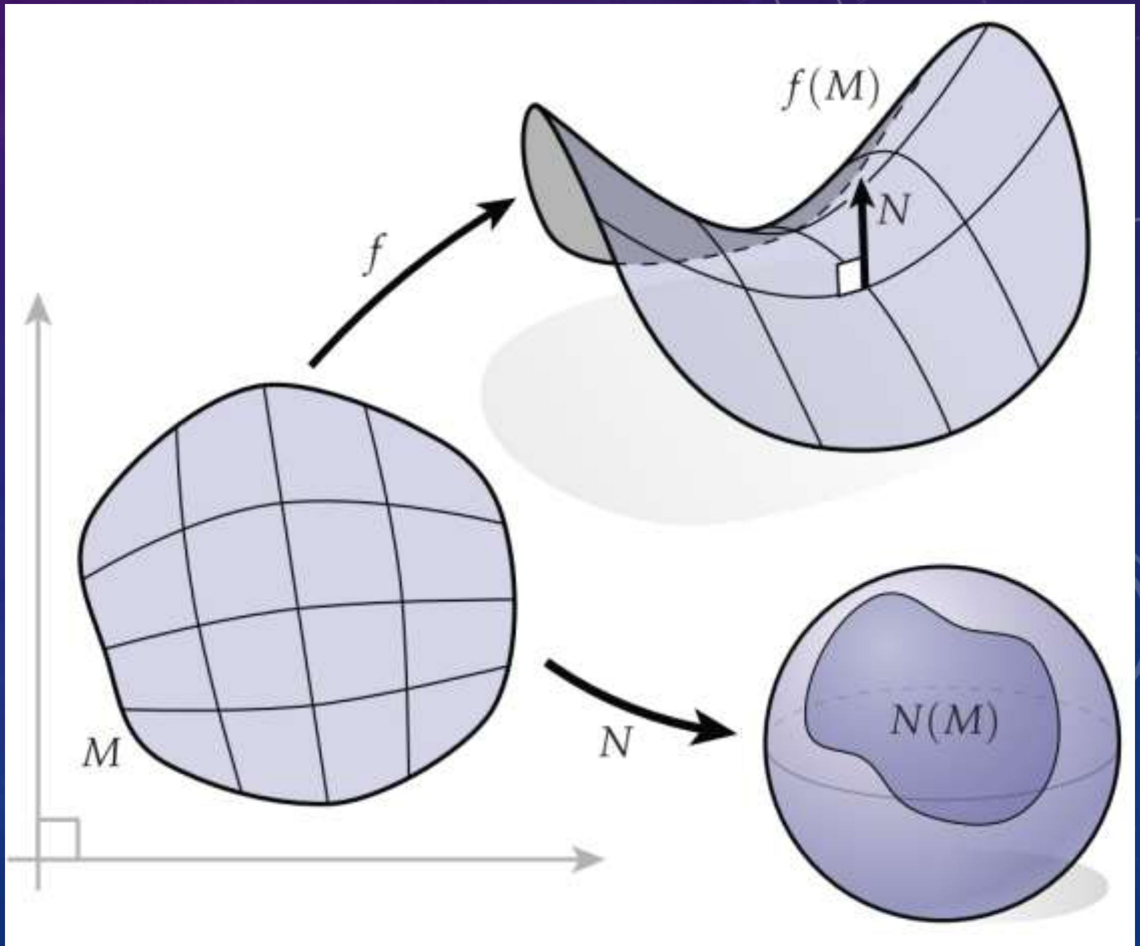


# Gaussian curvature and Spherical Area

As $df(X) \times S = dN(X)$, and $K = k_1 k_2 = |S|$

The area of Gauss map:

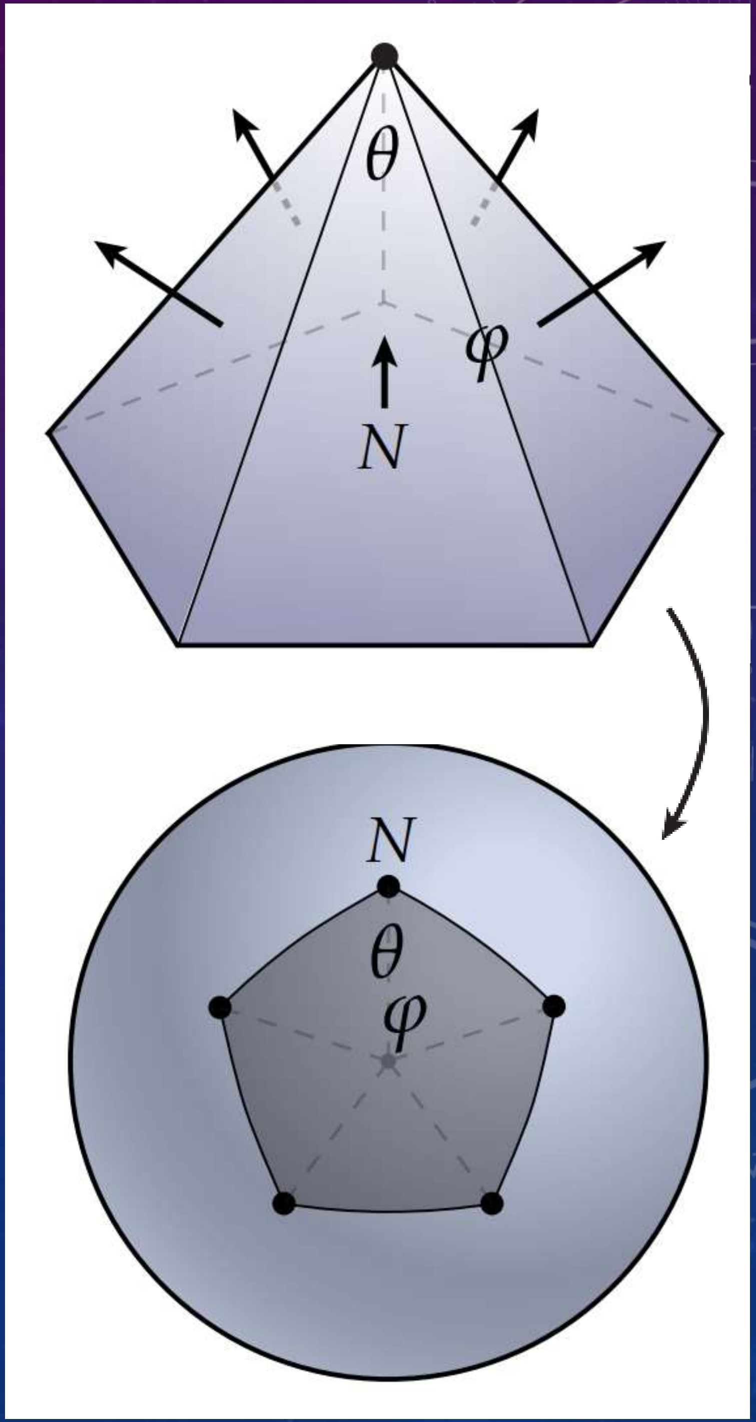
$$\int |dN| d\mathcal{U} = \int |S||df| d\mathcal{U} = \int K dA$$



# Angle Defect and Spherical Area

Consider the discrete Gauss map:

- unit normals on surface become points on the sphere
- dihedral angles on surface become interior angles on sphere
- interior angles on surface become dihedral angles on the sphere
- angle defect on surface becomes area on the sphere



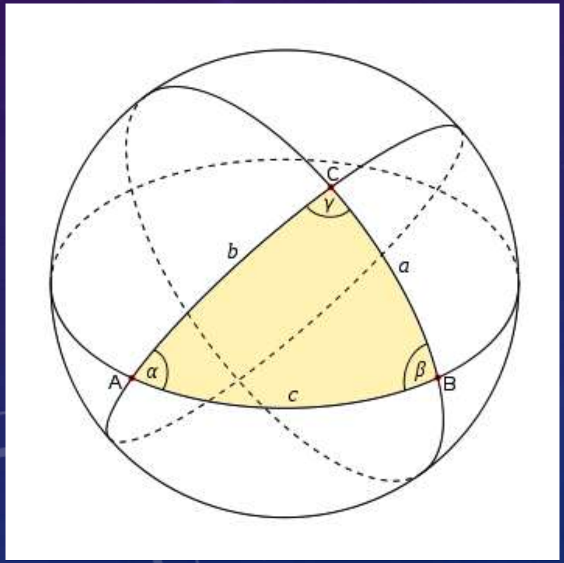
# Angle Defect and Spherical Area

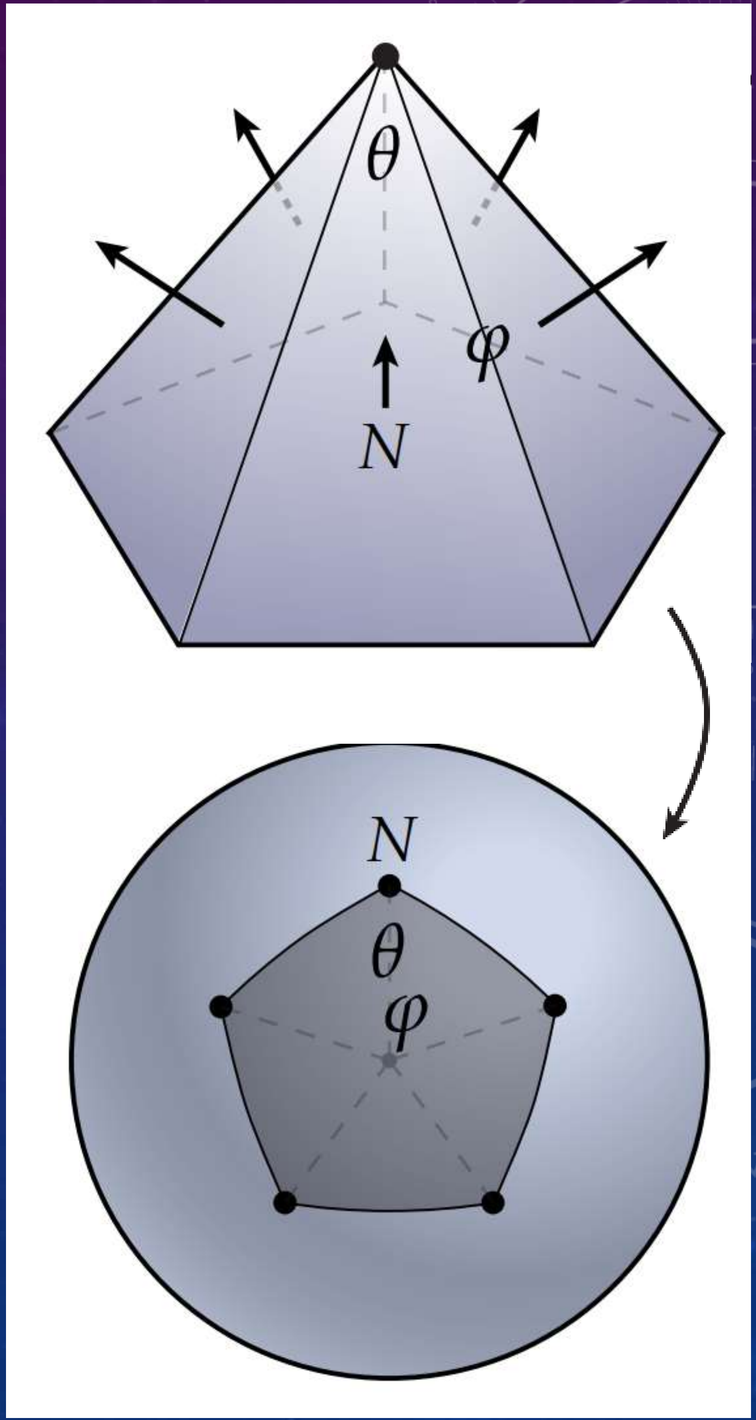
Spherical triangle area formula:

$$A = R^2(\alpha + \beta + \gamma - \pi)$$

↓

$$\text{Area(poly)} = 2\pi - \sum_{ijk} \theta_i^{jk} = \Omega_i$$



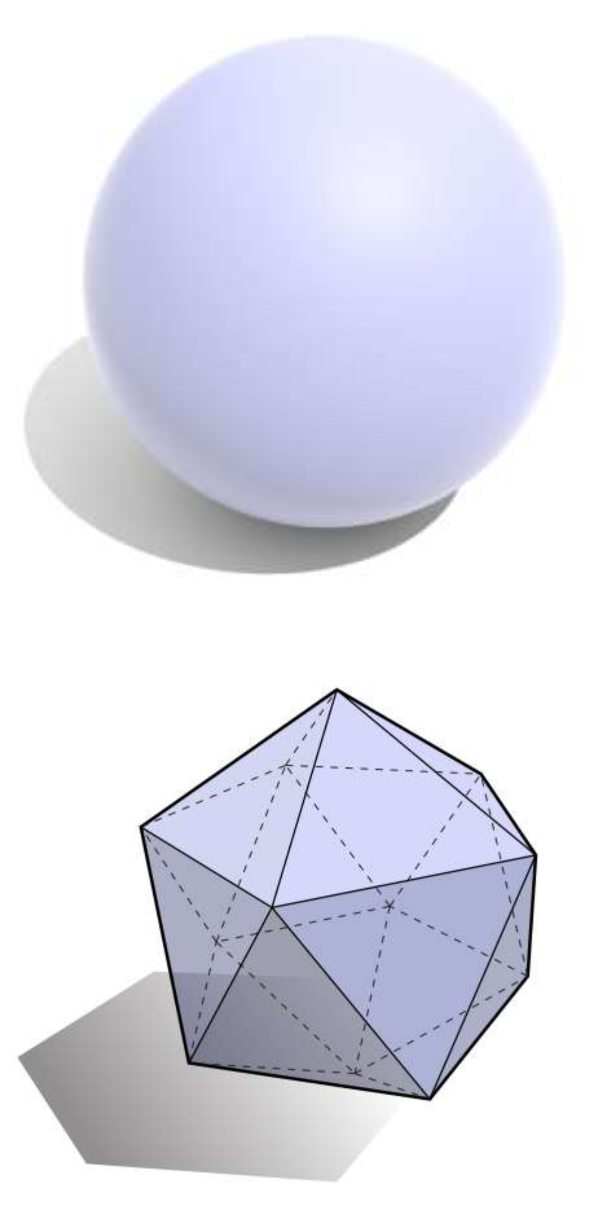
# Discrete Gauss Bonnet Theorem

Theorem. For a smooth surface of genus g, the total Gauss curvature is

$$\int_M K dA = 2\pi\chi$$

Theorem. For a discrete surface of genus g, the total angle defect is

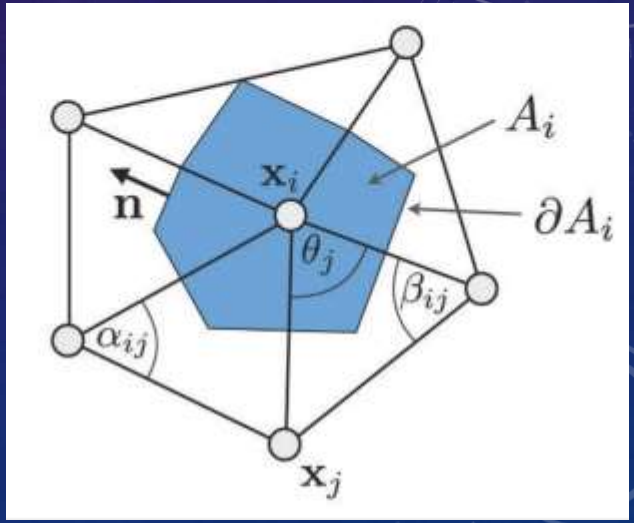
$$\sum_{i\in V} \Omega_i = 2\pi\chi$$



# Mean curvature

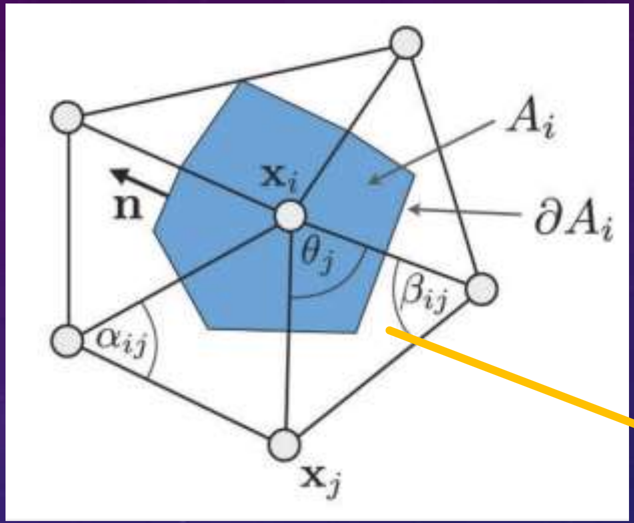
- For any smooth immersed surface $f$, $\Delta f = 2HN$
- Discretize $\Delta f$ on vertex neighbor

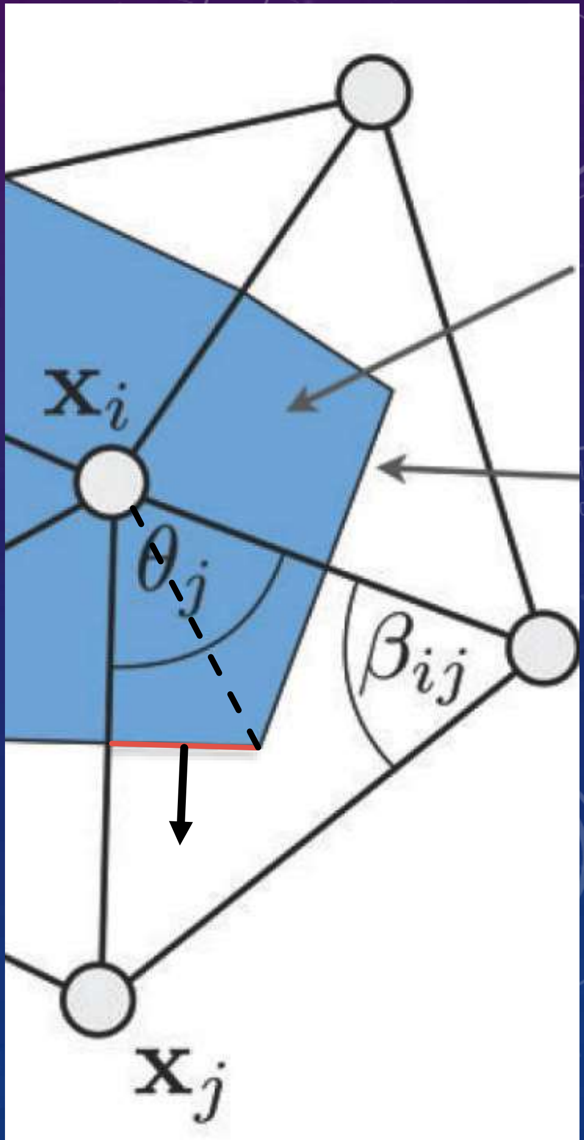
$$\int_{A_i} \Delta f dA = \int_{A_i} \nabla \cdot \nabla f dA = \int_{\partial A_i} \langle \nabla f, \boldsymbol{n} \rangle ds$$




- $A_i$ is the local averaging domain of vertex $i$.
- $\partial A_i$ is the boundary of $A_i$ .
- $\boldsymbol{n}$ is the outward pointing unit normal of the boundary.

# Mean curvature




- Consider red line segment,

$$\langle \nabla f, \boldsymbol{n} \rangle s = \left\langle \nabla f, \frac{X_i X_j}{\|X_i X_j\|} \right\rangle = (f_j - f_i)\frac{s}{\|X_i X_j\|} = \frac{\cot \beta_{ij}}{2}(f_j - f_i)$$

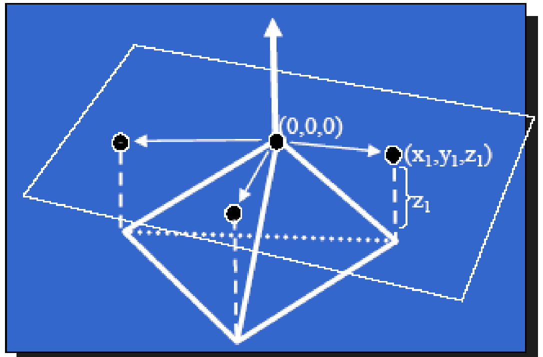
$$2H_i N_i = \int_{\partial A_i} \langle \nabla f, \boldsymbol{n} \rangle ds = \frac{1}{2}\sum\nolimits_{j \in \Omega(i)} (\cot \alpha_{ij} + \cot \beta_{ij})(f_j - f_i)$$

# Principal Curvatures

- Gaussian $K = \kappa_1 \kappa_2$, mean $H = \frac{\kappa_1 + \kappa_2}{2}$
- Principal : $\kappa_1 = H - \sqrt{H^2 - K}, \kappa_2 = H + \sqrt{H^2 - K}$
- Discrete principal: $\frac{H_i}{A_i} - \sqrt{(\frac{H_i}{A_i})^2 - \frac{K_i}{A_i}}$

# Principal directions

- Approximate surface by quadric
- At each mesh vertex (use surrounding triangles)
  - Compute normal at vertex
    - Typically average face normals
  - Compute tangent plane & local coordinate system
    - (node = (0,0,0))
  - For each neighbor vertex compute location in local system
    - relative to node and tangent plane




- Find quadric function approximating vertices

$$F(x,y,z) = ax^2 + bxy + cy^2 - z = 0$$

- To find coefficients use least squares fit

$$\min \sum_i (ax_i^2 + bx_iy_i + cy_i^2 - z_i)$$

- Given surface $F$ its principal curvatures $k_{min}$ and $k_{max}$ are real roots of:

$$k^2 - (a+c)k + ac - b^2 = 0$$

- *Mean curvature*: $H = (k_{min} + k_{max})/2$
- *Gaussian curvature:* $K = k_{min} k_{max}$

# Principal directions