Discrete Differential - Curves

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https://qingfang1208.github.io/

Applications of curves

- Geometry

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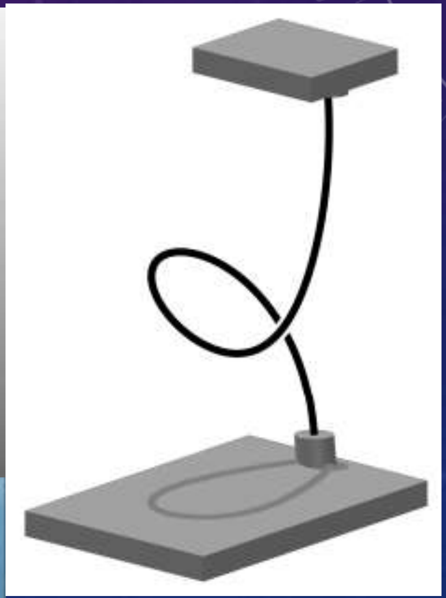
- Geometry



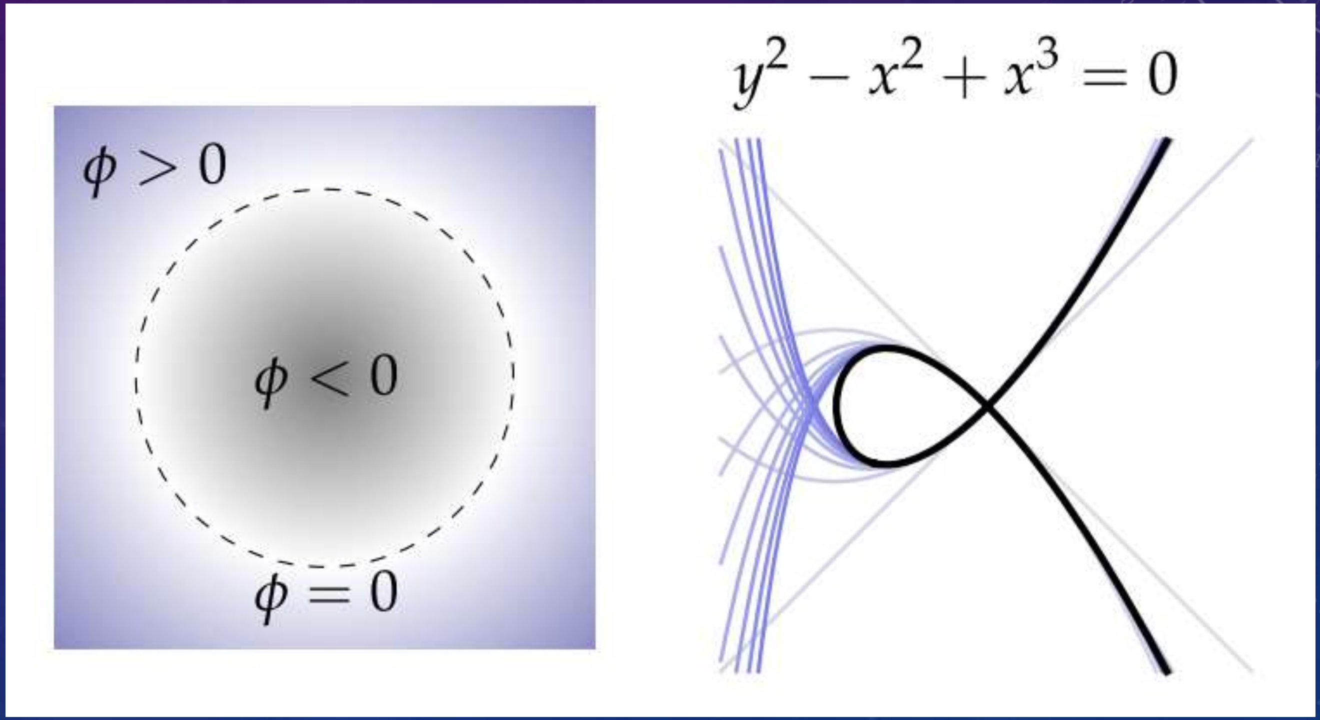
- Physics

Smooth Plane Curve

Representation of curves

- Implicit

$$x^2 + y^2 = 1$$



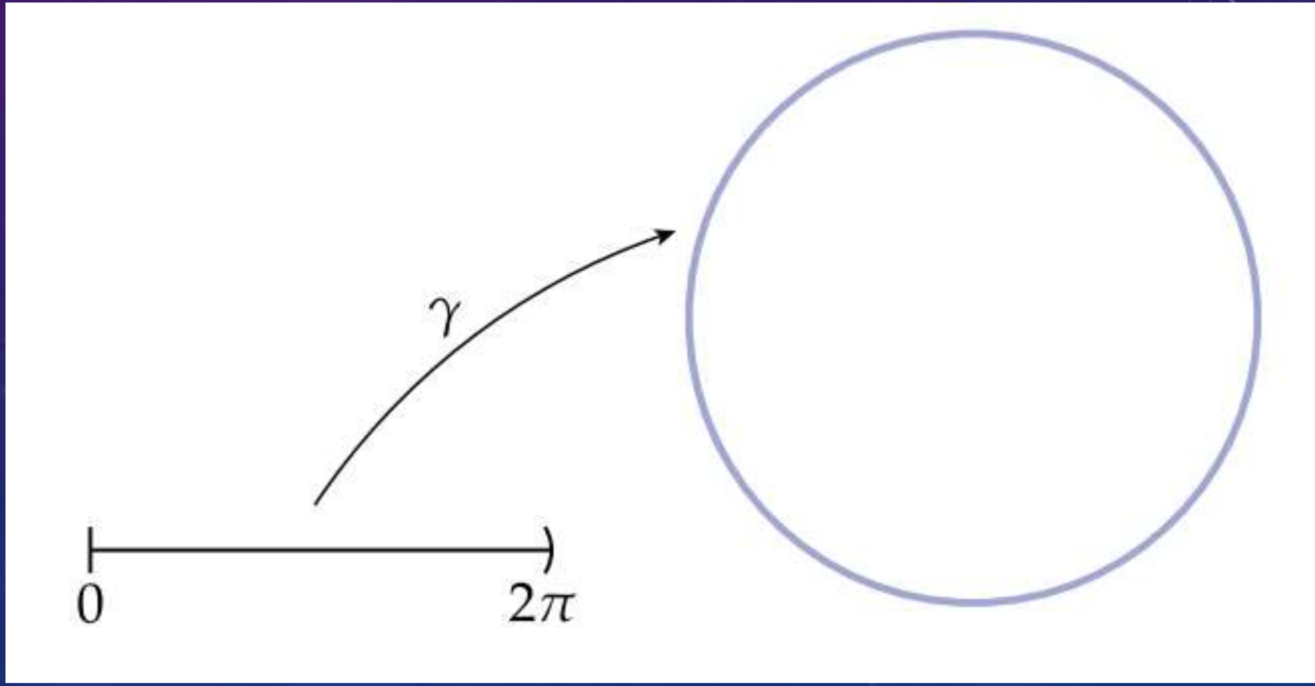
Representation of curves

- Implicit

$$x^2 + y^2 = 1$$

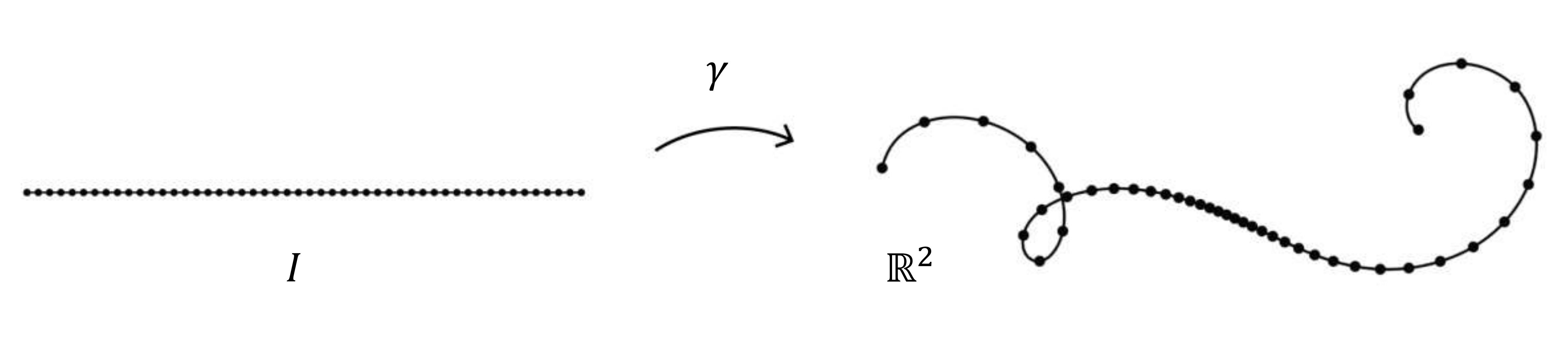
- Explicit

$$\begin{cases} x(\theta) = \cos\theta \\ y(\theta) = \sin\theta \end{cases}$$



Parameterized curve

- A parametrized plane curve is a continuous function $\gamma : I \rightarrow \mathbb{R}^2$, where the domain $I \subseteq \mathbb{R}$ is some (connected) interval on the real line.



Reparameterization

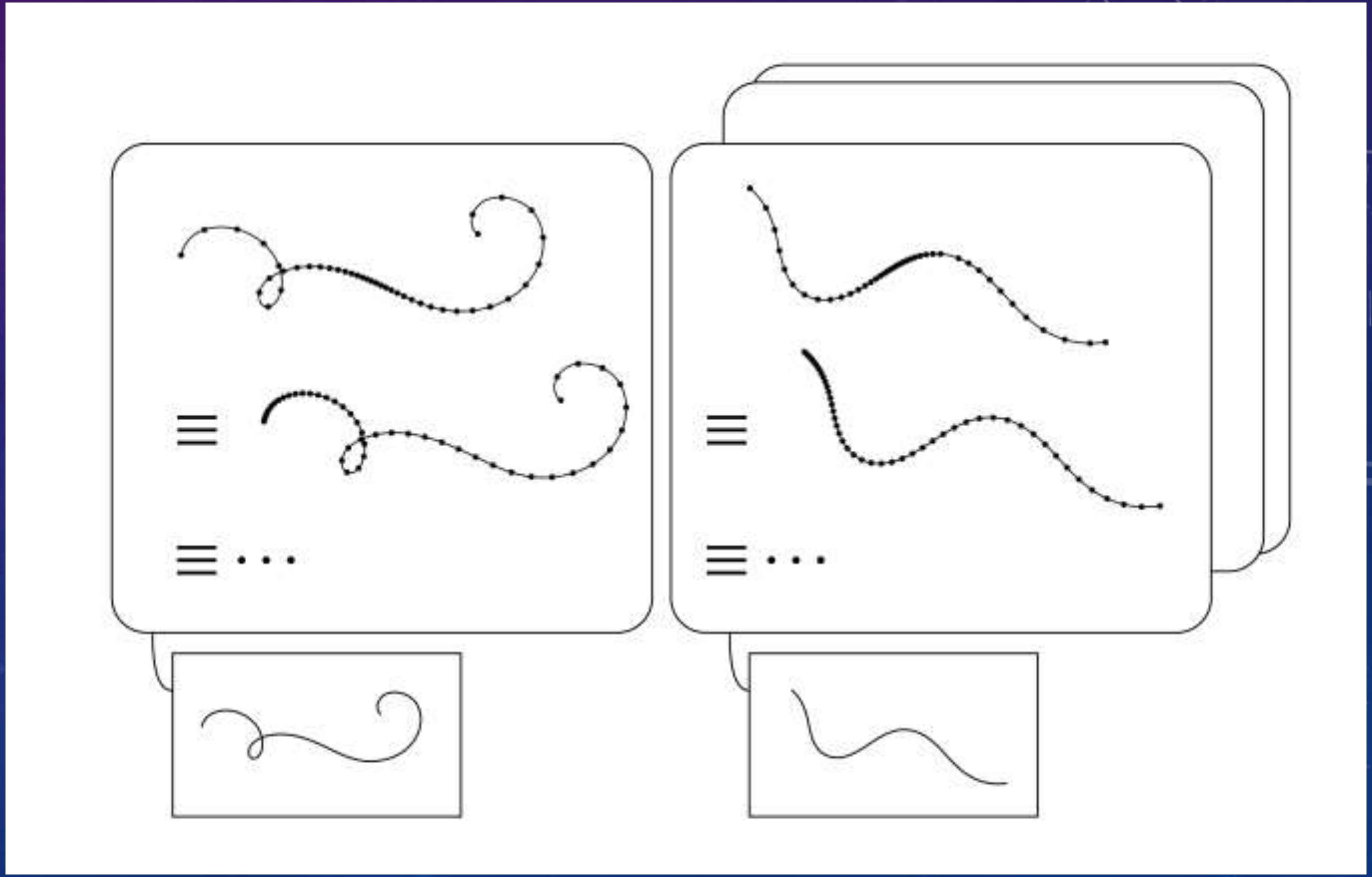
- Two parametrizations $\gamma_1 : I_1 \rightarrow \mathbb{R}^2$ and $\gamma_2 : I_2 \rightarrow \mathbb{R}^2$ are said to yield the same curve if there exists a continuous and continuously invertible function $\varphi: I_1 \rightarrow I_2$, called a reparameterization, such that $\gamma_1(t) = \gamma_2(\phi(t))$ for all $t \in I_1$; in short $\gamma_1 = \gamma_2 \circ \phi$

Reparameterization

$$\gamma_1 : I_1 \to \mathbb{R}^2$$

$$\gamma_2 : I_2 \to \mathbb{R}^2$$

$$\Rightarrow \gamma_1 = \gamma_2 \circ \phi$$

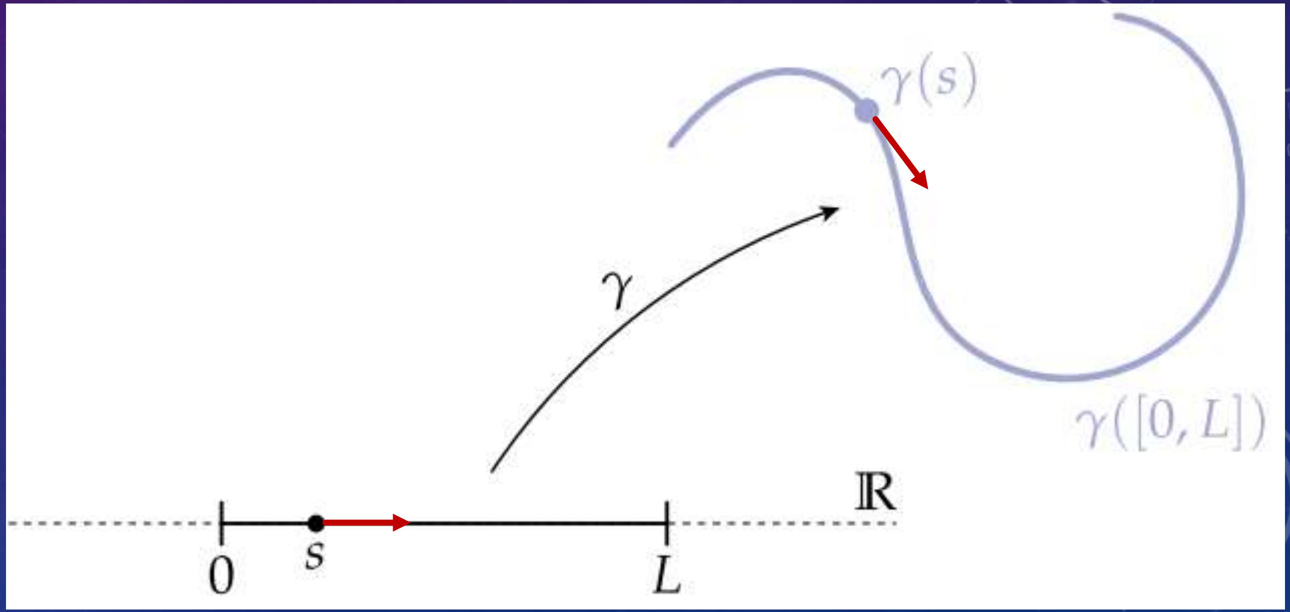


Differential of a curve

- $\gamma'(s) = \lim_{\delta \to 0} \frac{\gamma(s+\delta) - \gamma(s)}{\delta}$
- Tangent to a curve

Unit tangent: $T(s) = \frac{\gamma(s)}{|\gamma\prime(s)|}$

Arc length parameterized



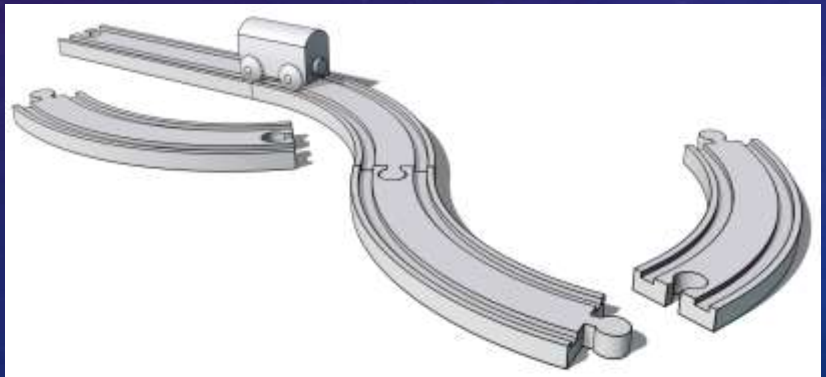
Regular curves

- A parametrized curve $\gamma : I \to \mathbb{R}^2$ is regular if γ has continuous first derivative, and has non-vanishing speed $|\gamma'(s)| \neq 0$ for all $s \in I$.
- Every regular parametrized curve $\gamma : I \to \mathbb{R}^2$ can be reparametrized by its arclength.

$$I = [a, b], L = \int_a^b |\gamma'(t)| dt \Rightarrow \phi: [0, L] \to [a, b], \phi(s) = a + \int_0^s \frac{1}{|\gamma'(t)|} dt$$

k-th order smooth curves

- A regular parametrized curve is k-th order smooth if γ has continuous k-th derivative, write as $C^k(I, \mathbb{R}^2)$. If in addition $|\gamma'| \neq 0$, then γ is called a C^k regular parameterized curve.



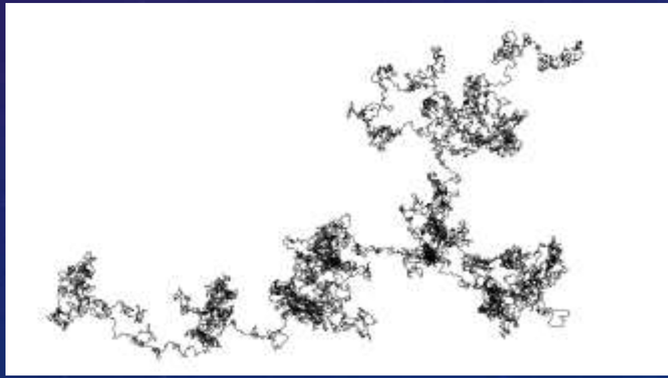
$C^1(I, \mathbb{R}^2), |\gamma'| \neq 0$

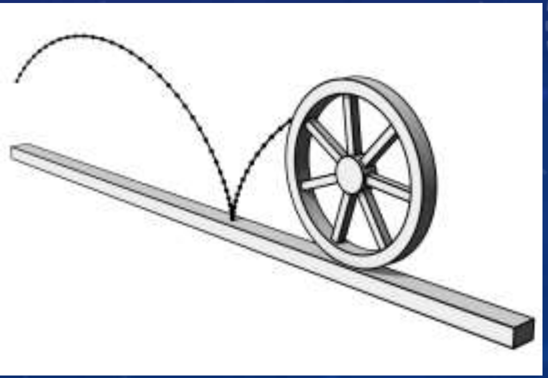
$C^2(I, \mathbb{R}^2), |\gamma'| \neq 0$

Irregular $C^k(I, \mathbb{R}^2)$

- $y = x^2 \to \gamma(s) = (s, s^2)$

- $y = |x| \to \tilde{\gamma}(s) = \begin{cases} (s^3, s^3), s \geq 0 \\ (s^3, -s^3), s < 0 \end{cases}$



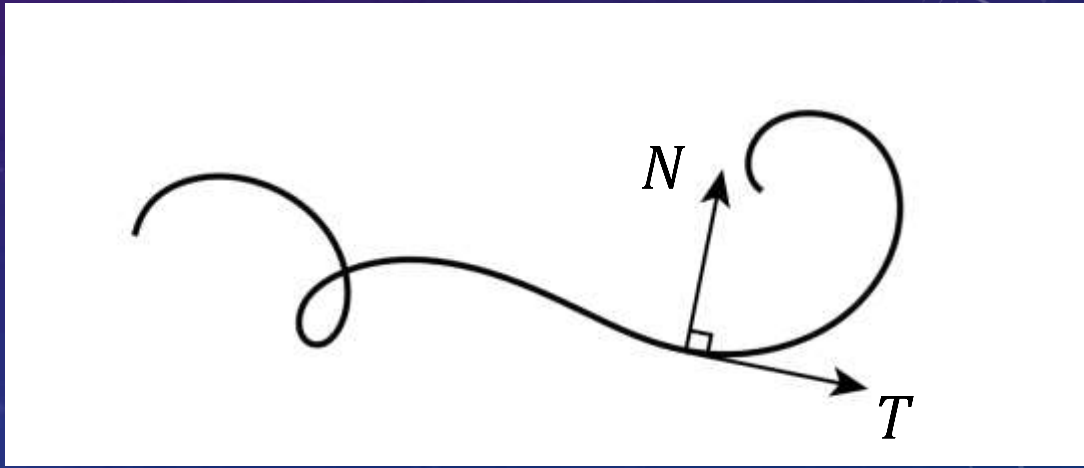


Curvature

- The rate of change of T

$$\langle T, T \rangle = 1 \Longrightarrow \left\langle \frac{dT}{ds}, T \right\rangle = 0$$

$$\kappa = \left\langle \frac{dT}{ds}, N \right\rangle$$



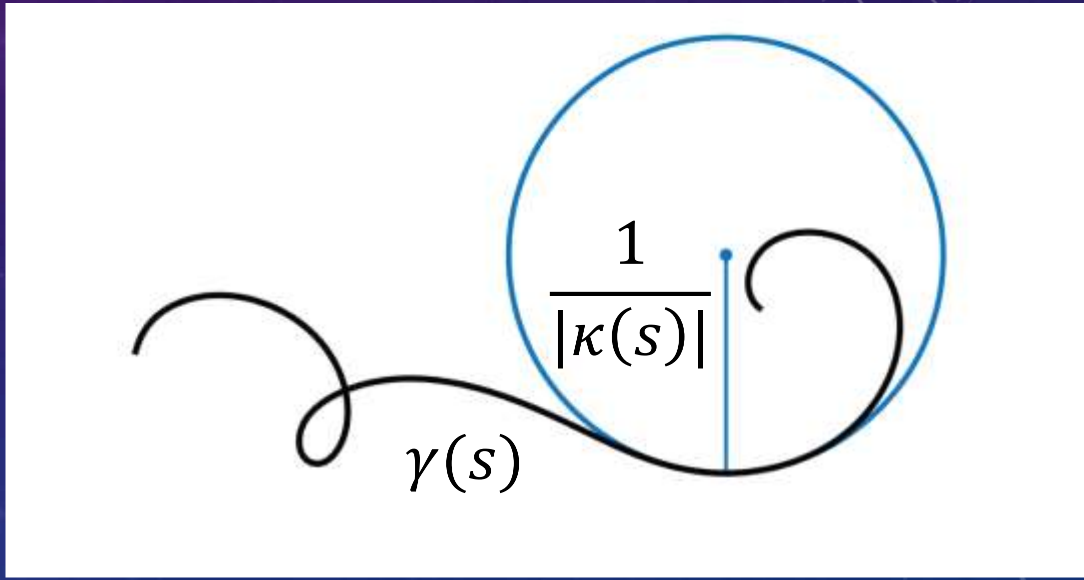
Osculating circle

- Circle: $\gamma(s) = \begin{bmatrix} R\cos(s/R) \\ R\sin(s/R) \end{bmatrix}$

$$T(s) = \begin{bmatrix} -\sin(s/R) \\ \cos(s/R) \end{bmatrix}$$

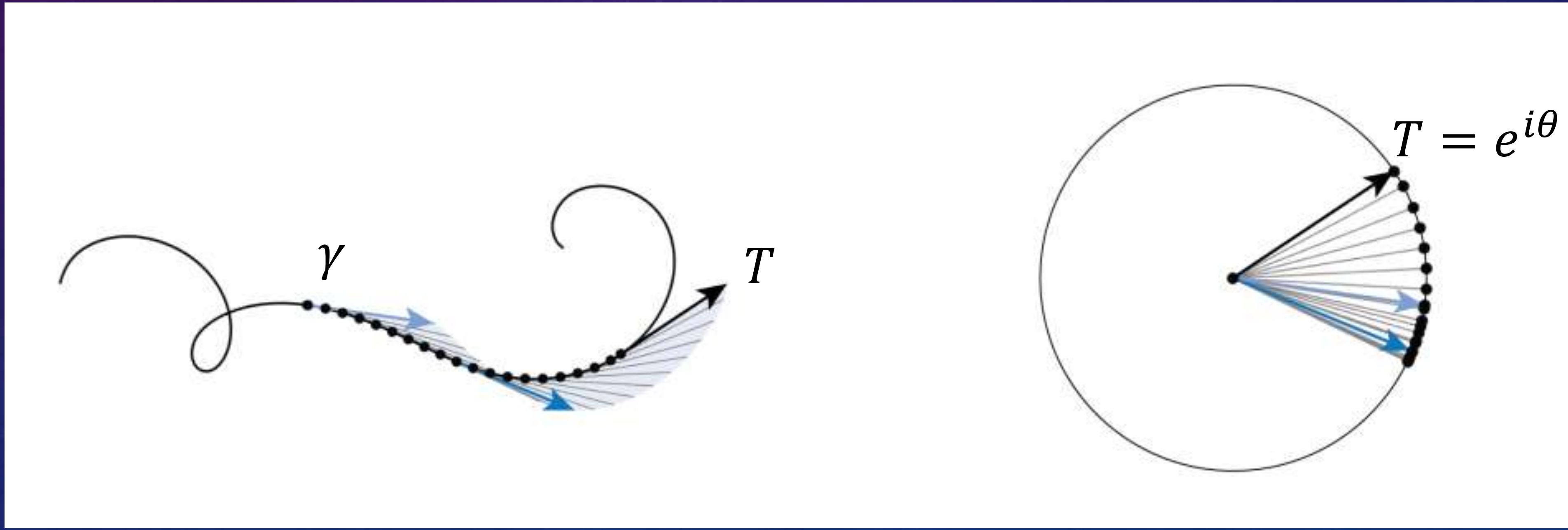
$$N(s) = \begin{bmatrix} -\cos(s/R) \\ -\sin(s/R) \end{bmatrix}$$

$$\Longrightarrow T(s) = \frac{1}{R} N(s)$$



Turning angle

As $T(s) = \cos\theta(s) + i\sin\theta(s) = e^{i\theta(s)}$, we can define $\theta : I \to \mathbb{R}\ mod\ 2\pi$



Turning angle

As $T(s) = \cos\theta(s) + i\sin\theta(s) = e^{i\theta(s)}$, we can define $\theta: I \to \mathbb{R}\ mod\ 2\pi$

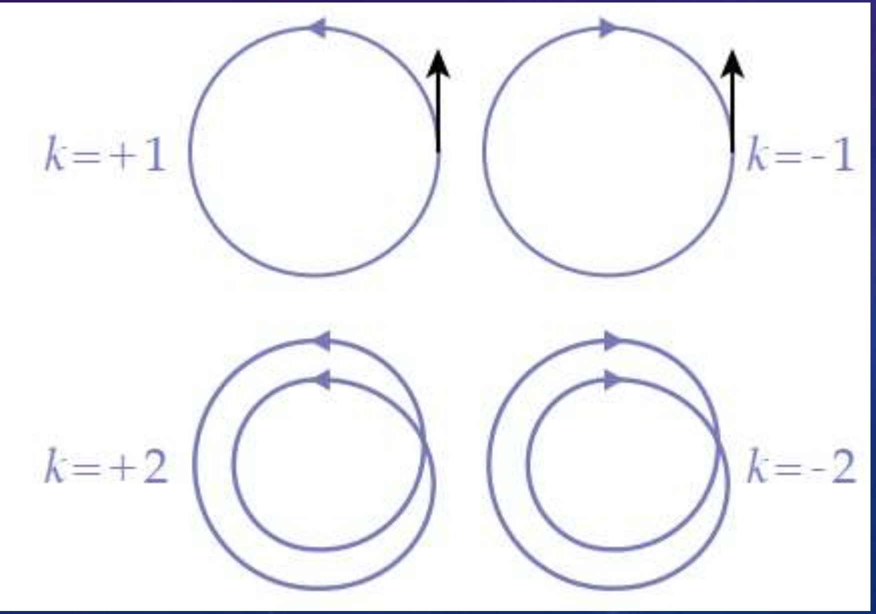
$$\frac{d}{ds}T = \frac{d}{ds}e^{i\theta(s)} = i\frac{d\theta}{ds}e^{i\theta} = \frac{d\theta}{ds}N \Rightarrow \kappa = \frac{d\theta}{ds}$$

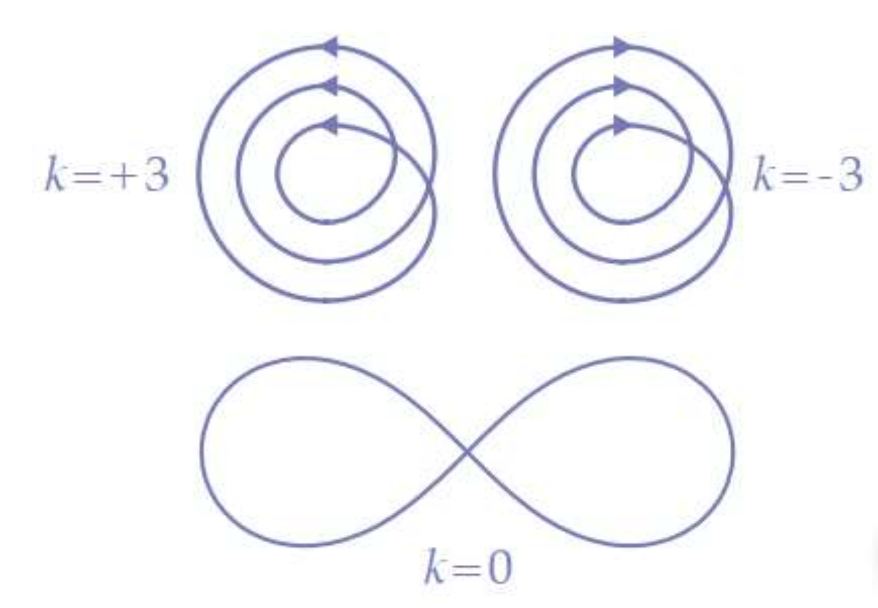
$$\theta(b) - \theta(a) = \int_a^b \kappa(s)ds$$

Turning number

The turning angle $T(s)$ of a close curve is always multiples of 2π. We can define

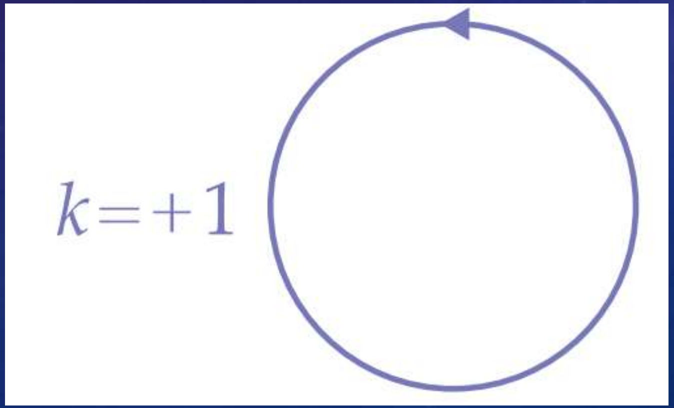
turning number : $k = \frac{T(L)-T(0)}{2\pi}$

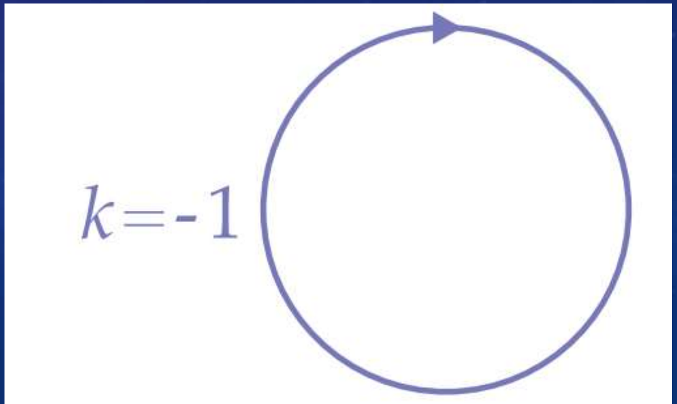




Whitney-Graustein Theorem

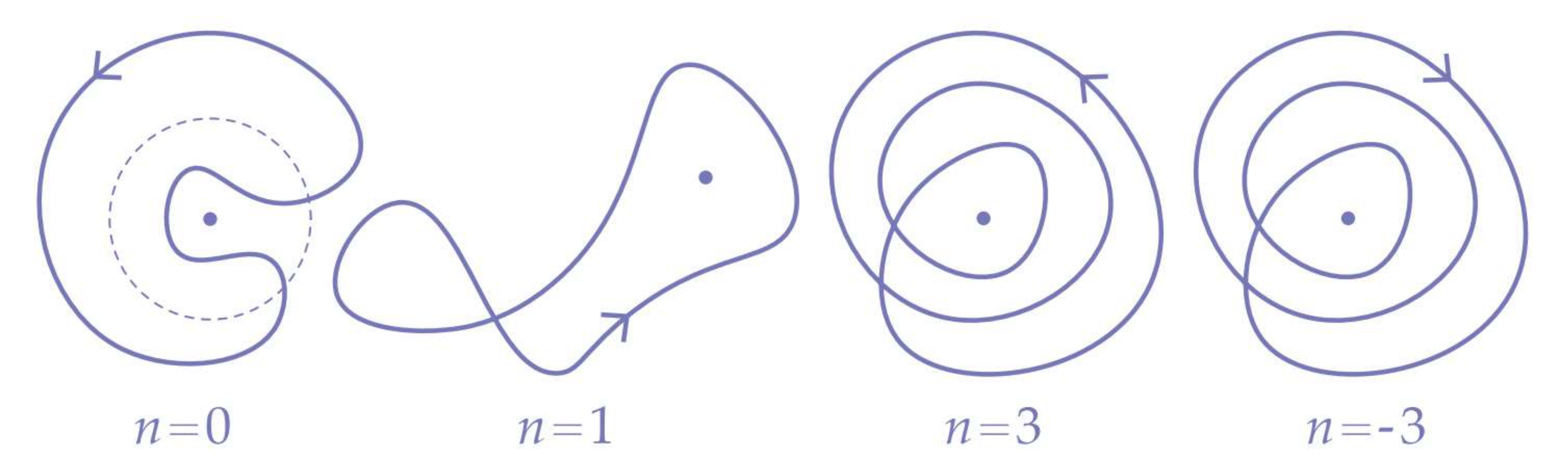
- Two curves are related by regular homotopy if one can continuously “deform” one into the other while remaining regular (immersed).
- Theorem : Two curves have the same turning number k if and only if they are regularly homotopic.





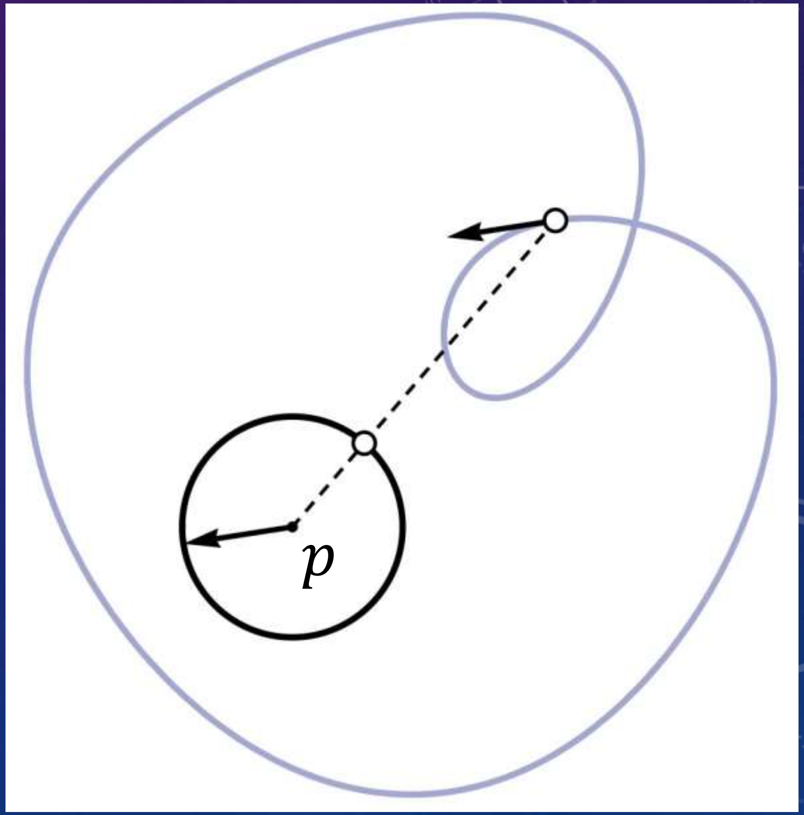
Winding number

Winding number n is the number of times the curve "goes around" a particular point p



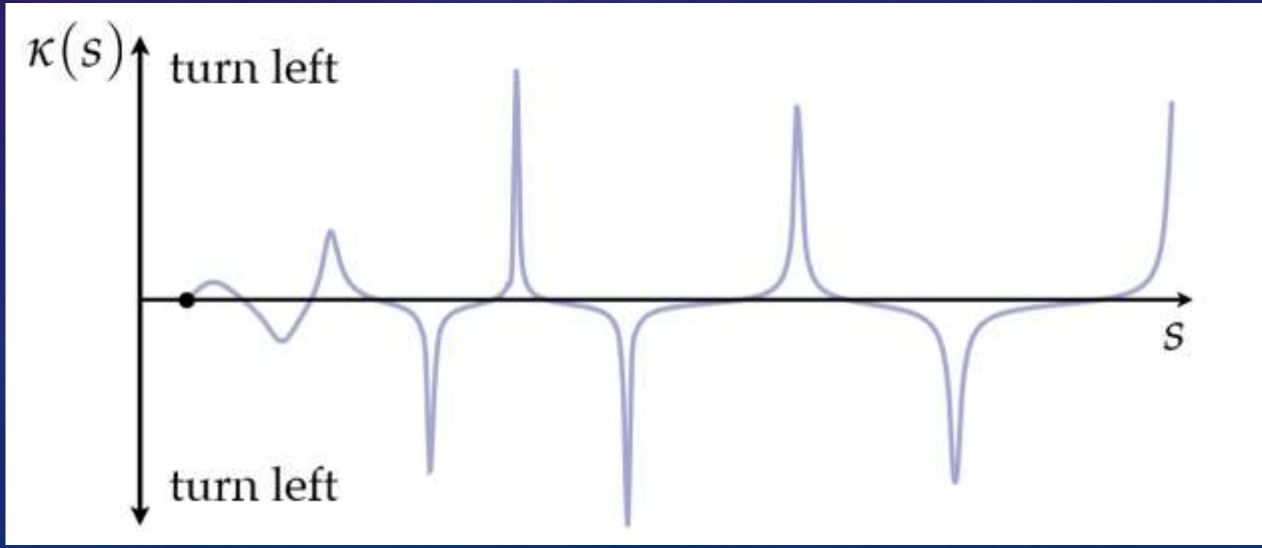
Turning number and winding number

- Turning number k is the winding number of tangent curve $T(s)$ around original point.
- Winding number n is turning number of a new curve obtained by projecting the curve onto the circle around p



Fundamental theorem of plane curves

- Up to rigid motions, an arclength parameterized plane curve is uniquely determined by its curvature.
- Curvature tells us how to "steer" as we move at unit speed.



Recovering a curve from curvature

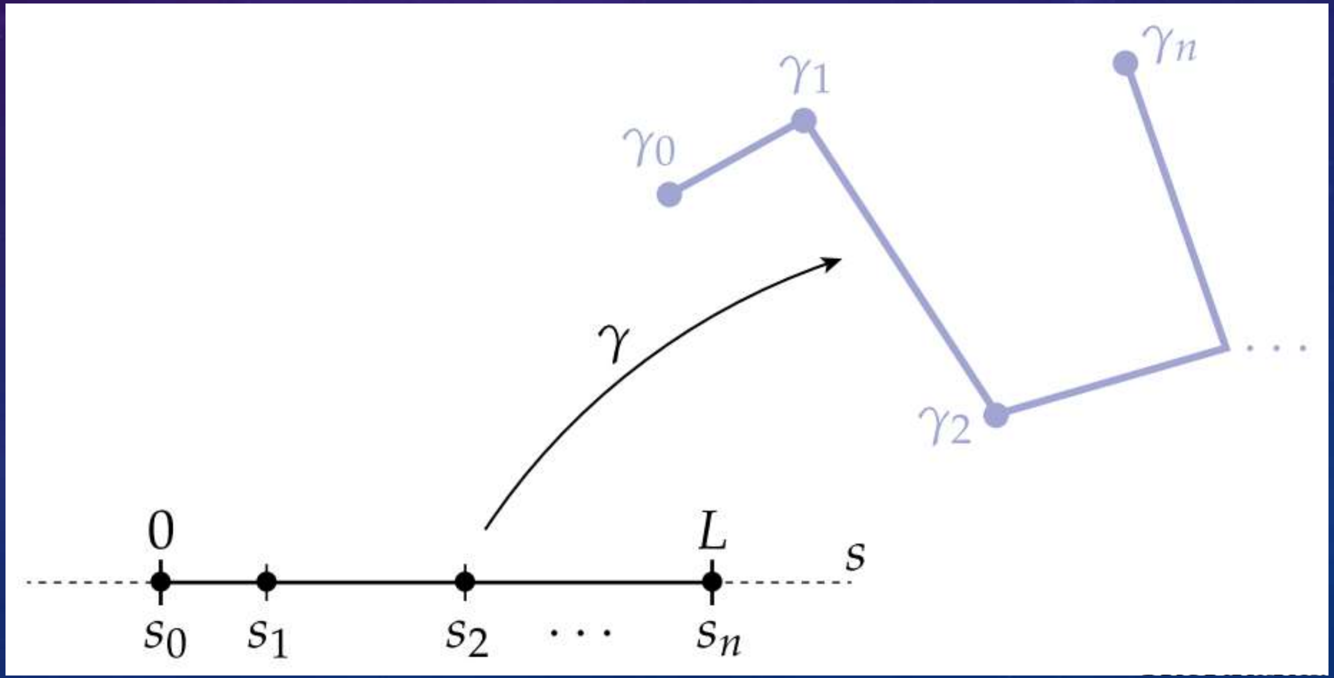
- Given only the curvature function, how can we recover the curve?

1. $\theta(s) = \theta(0) + \int_a^b \kappa(t)dt$
2. $T(s) = (\cos\theta(s), \sin\theta(s))$
3. $\gamma(s) = \int_0^s T(t)dt$

Discrete Plane Curve

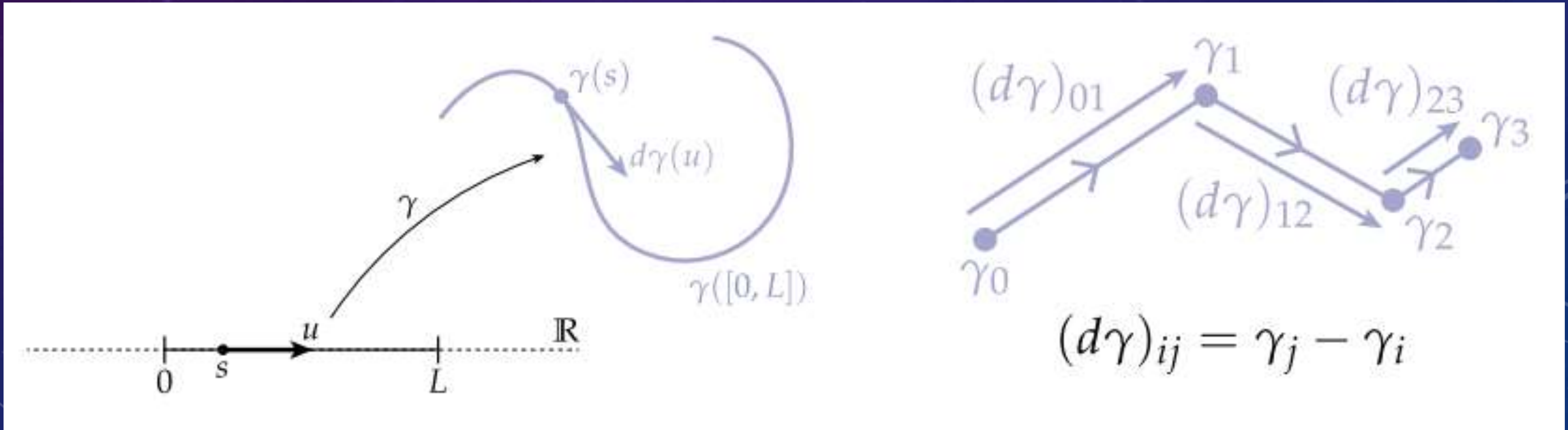
Polygonal curve

- A discrete plane curve, or more precisely a polygonal curve is a map $\gamma : I \to \mathbb{R}^2$ where I is the ordered index set $I = (s_0, s_1, \dots, s_{n-1}, s_n)$



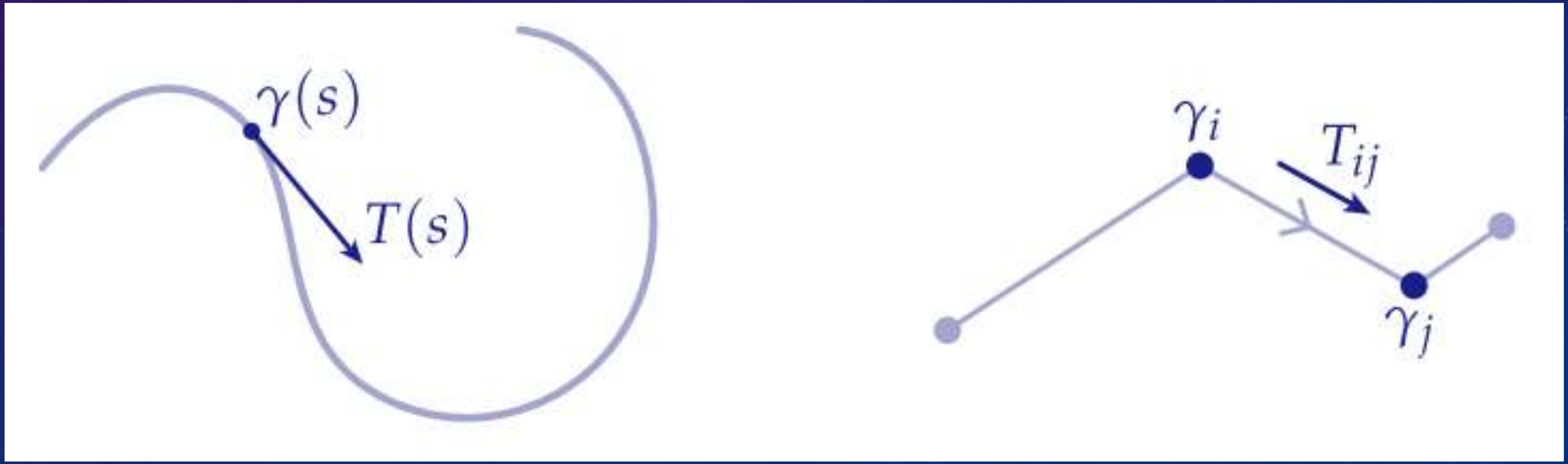
Discretization

- Differential → edge vector: $(d\gamma)_{ij} = \gamma_j - \gamma_i$



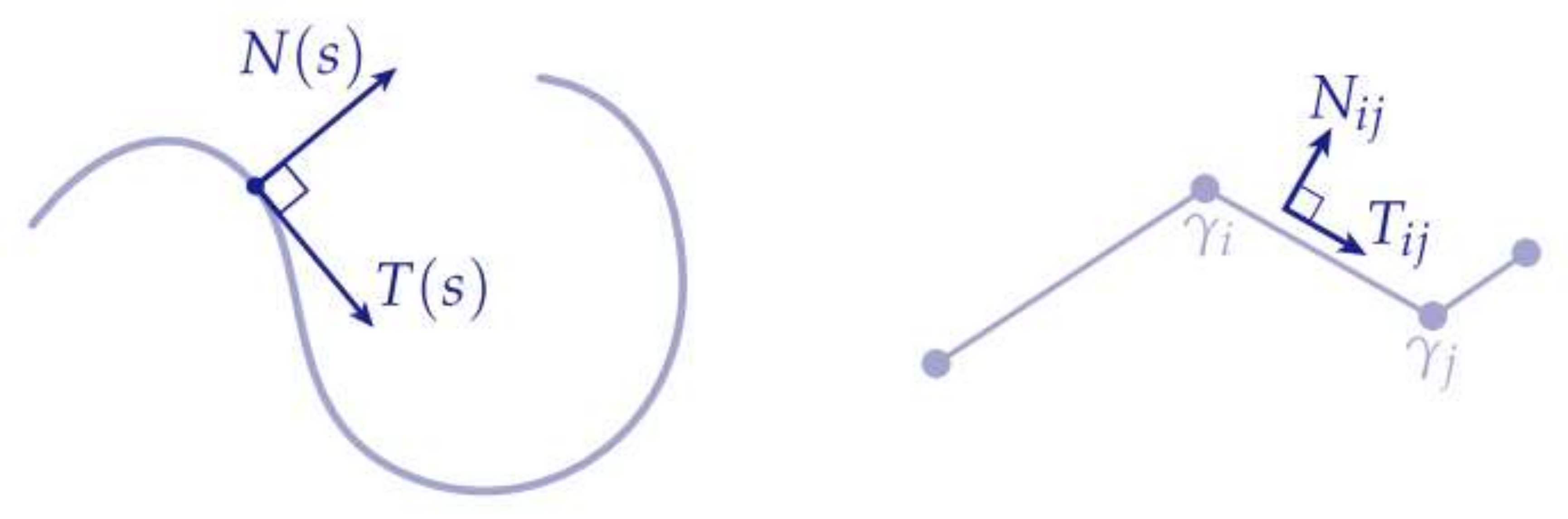
Discretization

- Differential → edge vector: $(d\gamma)_{ij} = \gamma_j - \gamma_i$
- Discrete tangent vector: $T_{ij} = (d\gamma)_{ij}/|(d\gamma)_{ij}|$



Discretization

- Differential → edge vector: $(d\gamma)_{ij} = \gamma_j - \gamma_i$
- Discrete tangent: $T_{ij} = (d\gamma)_{ij}/|(d\gamma)_{ij}|$. Discrete normal: $N_{ij} = iT_{ij}$

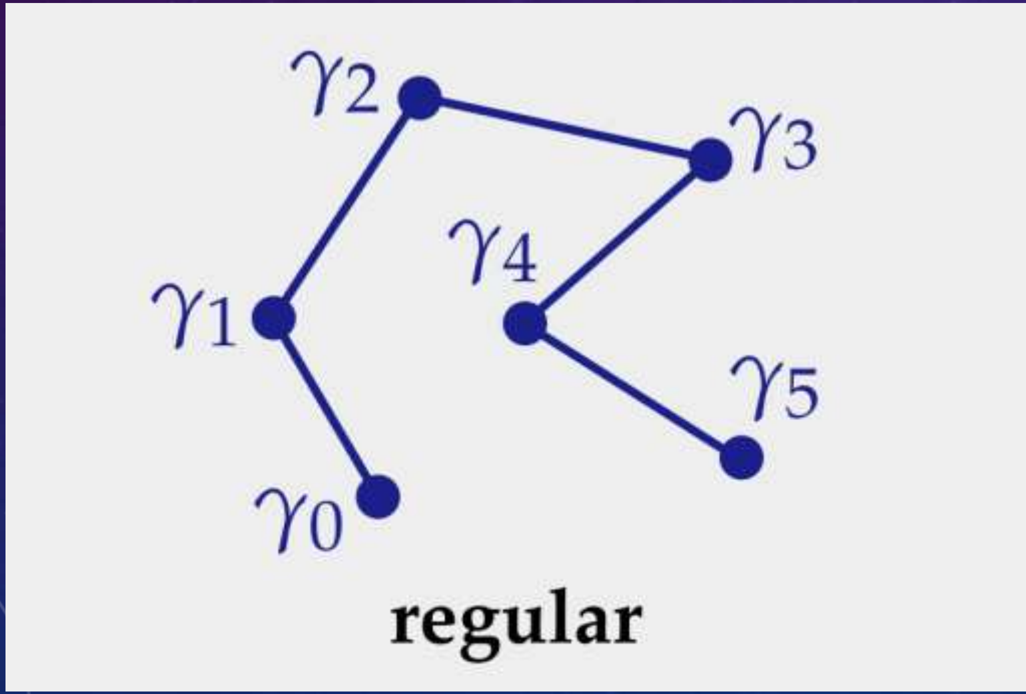


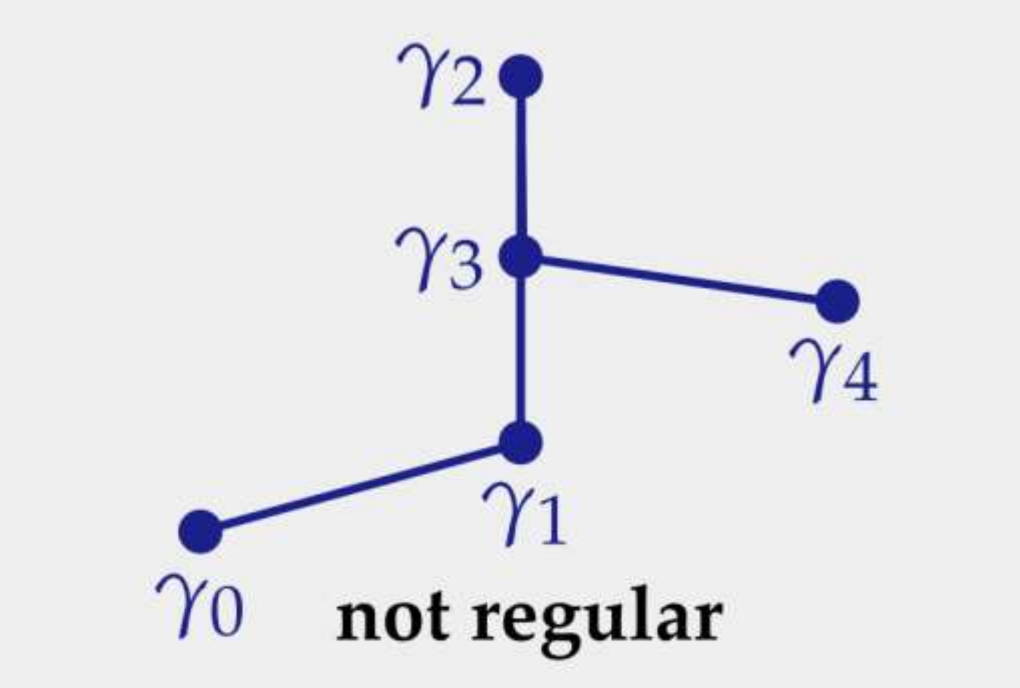
Discretization

- Differential → edge vector: $(d\gamma)_{ij} = \gamma_j - \gamma_i$
- Discrete tangent: $T_{ij} = (d\gamma)_{ij}/|(d\gamma)_{ij}|$. Discrete normal: $N_{ij} = iT_{ij}$
- Arc length parametrized: $\left|(d\gamma)_{ij}\right| \equiv c > 0$ for all edges.

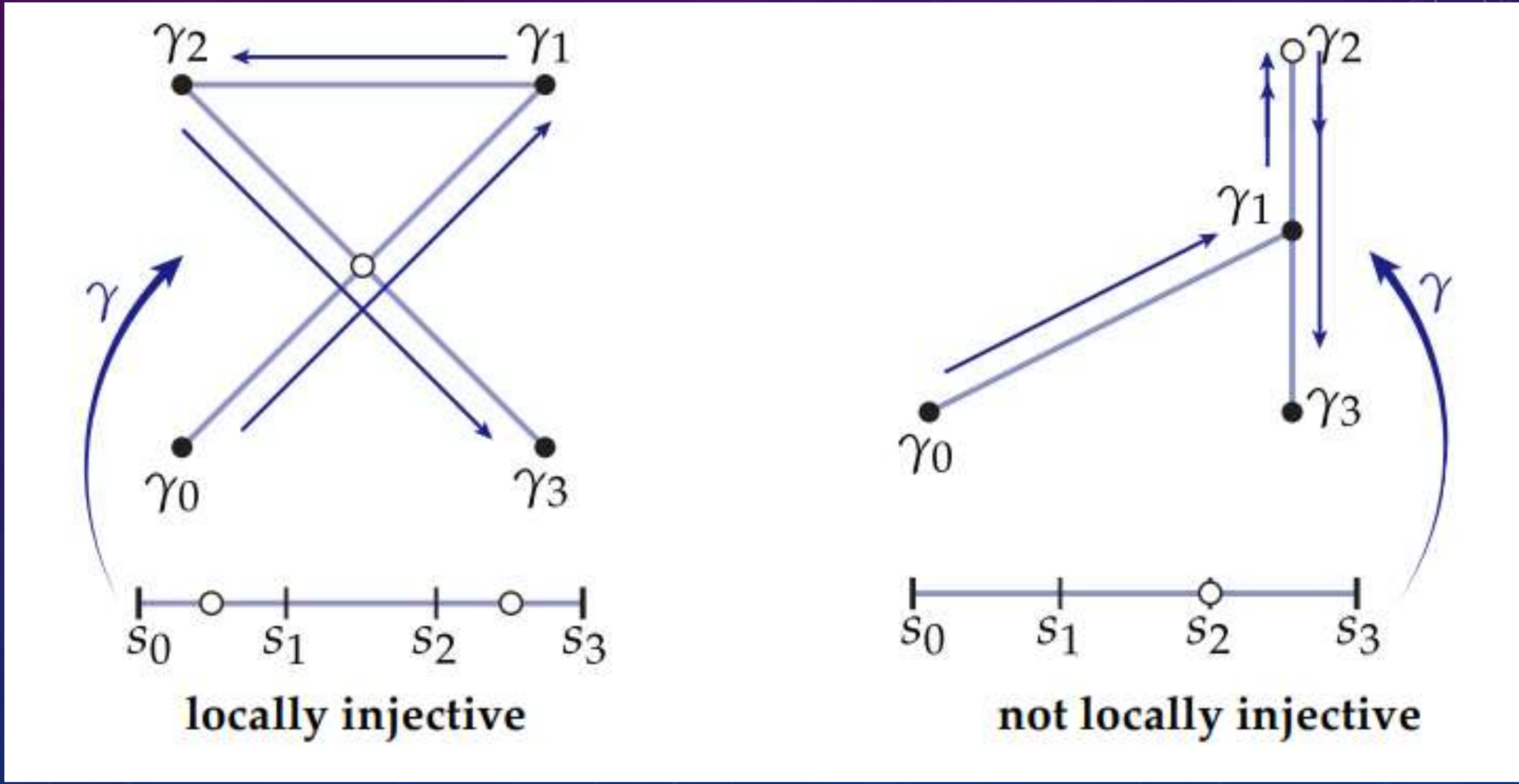
Regular discrete curve

- A polygonal curve is called regular if $\left|(d\gamma)_{ij}\right| \neq 0$ and $T_{ij} \neq -T_{ji}$ for all edges.



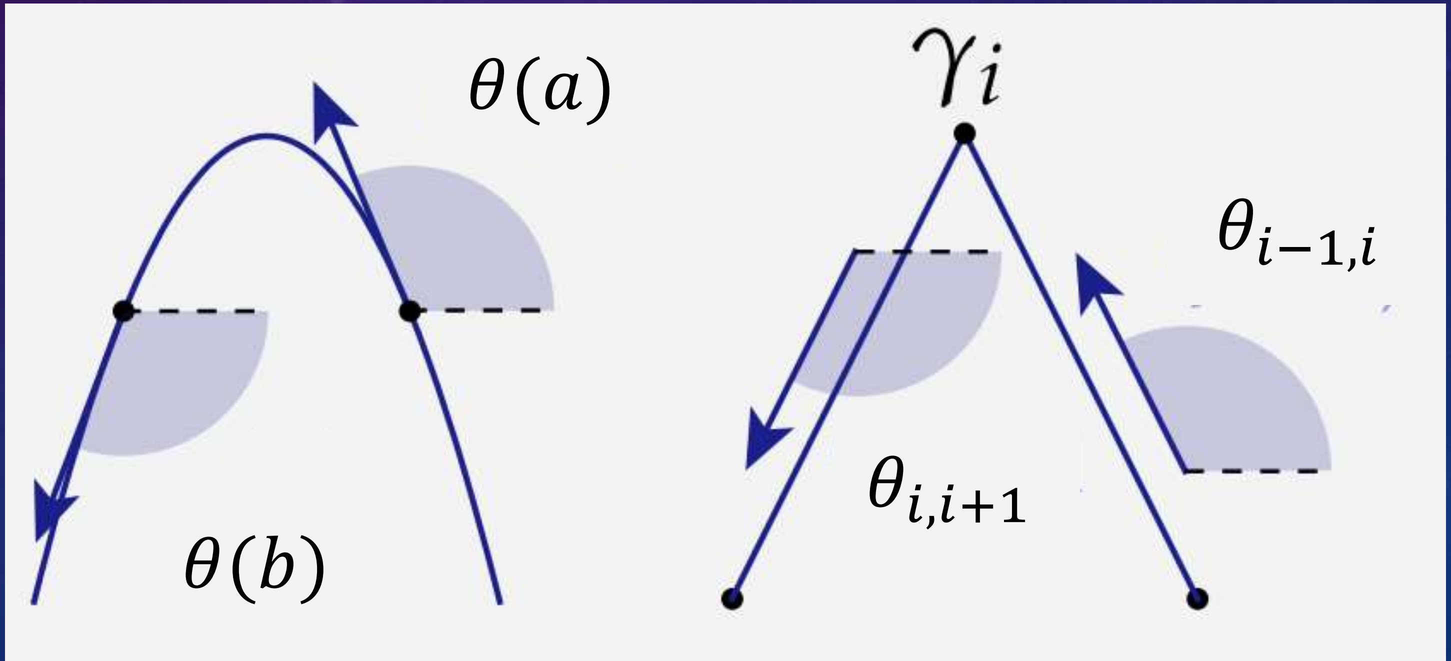


Regular discrete curve ⟺ locally injective map



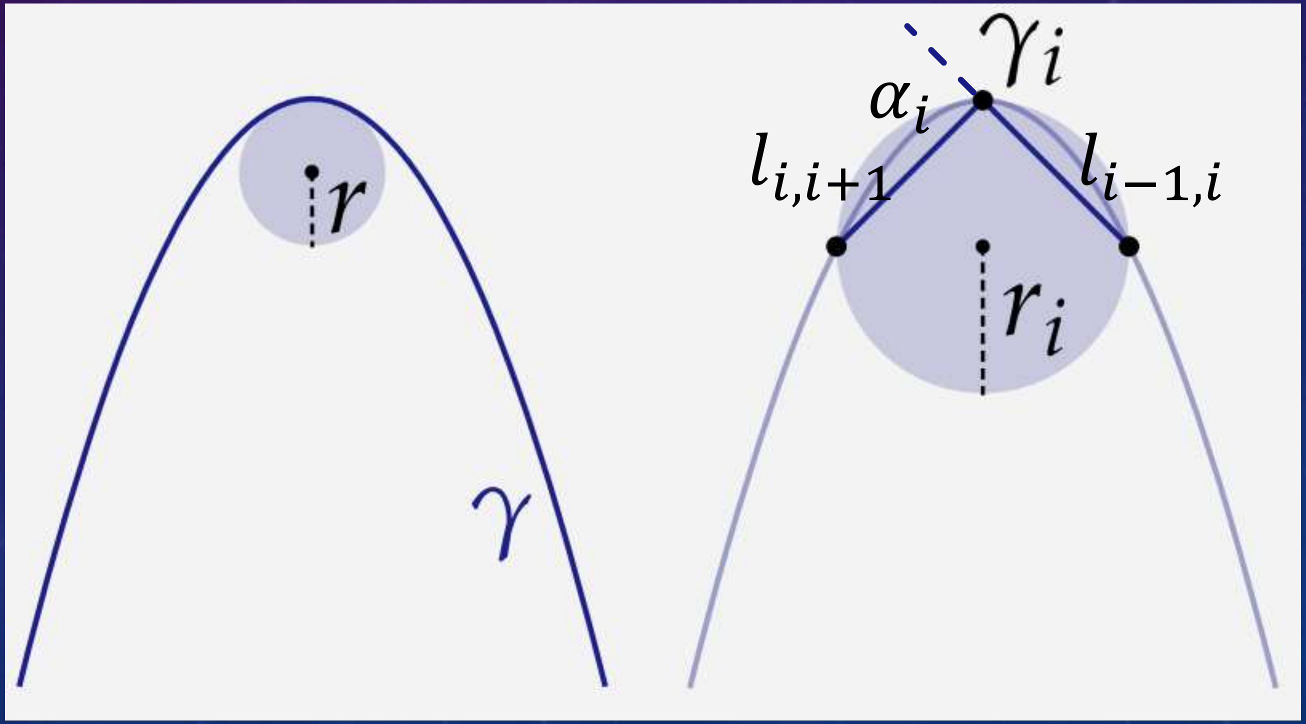
Discrete curvature

- $\theta(b) - \theta(a) = \int_a^b \kappa(s)ds \Longrightarrow$ vertex exterior angle (turning angle)



Discrete curvature

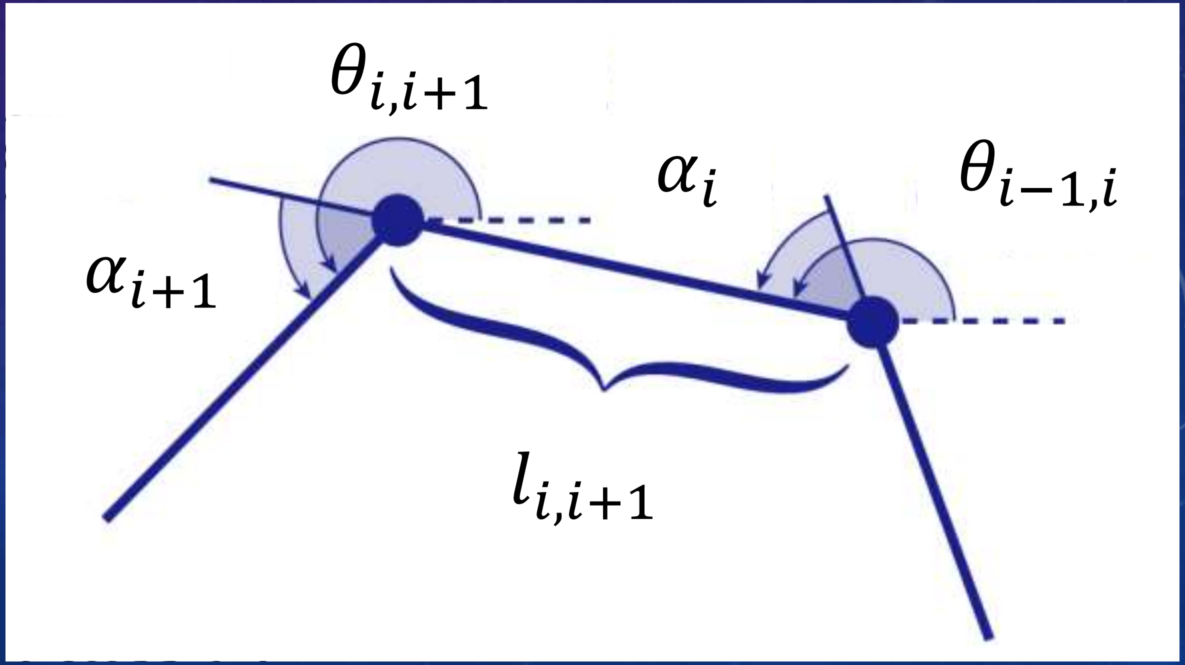
- Osculating circle $\Longrightarrow$ vertex osculating circle



Fundamental theorem of discrete plane curves

- Up to rigid motions, a regular discrete plane curve is uniquely determined by its edge lengths and turning angles.

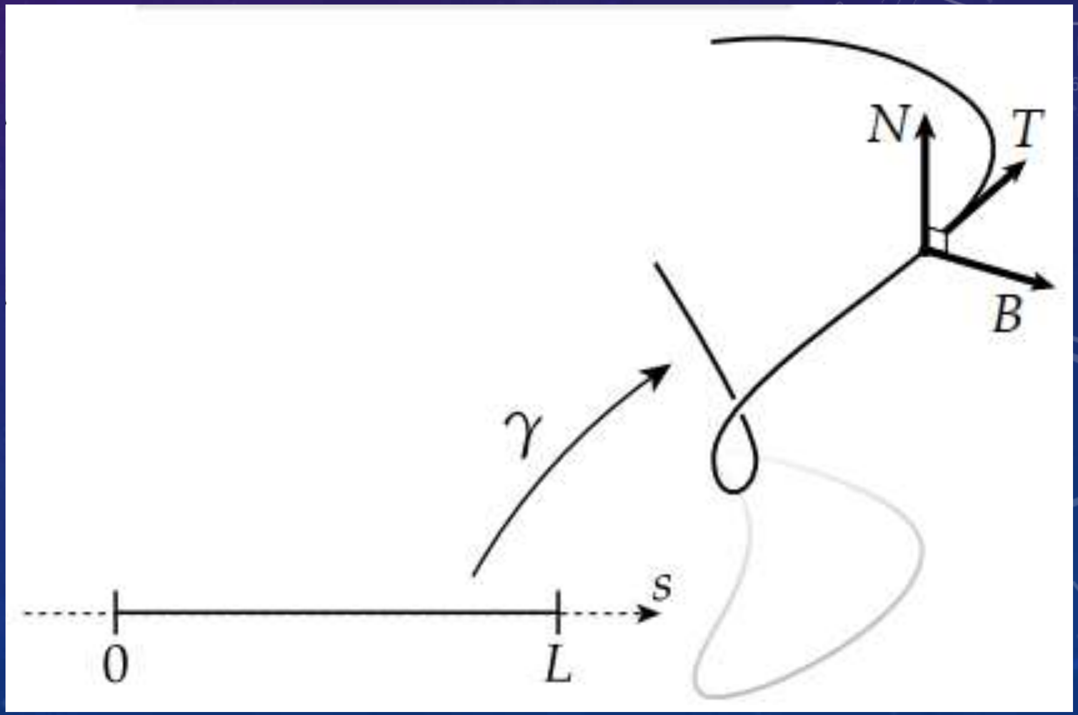
1. $\theta_{i,i+1} = \theta_{0,1} + \sum_{k=1}^{i} \alpha_k$
2. $T_{i,i+1} = (\cos\theta_{i,i+1}, \sin\theta_{i,i+1})$
3. $\gamma_i = \gamma_0 + \sum_{k=1}^{i} T_{k-1,k}$



Smooth Space Curve

Frame of curves

- A function $N\colon I \to \mathbb{S}^2$ satisfying $\langle N, T \rangle \equiv 0$ is a normal vector field.
- $B = T \times N$ is called binormal vector field.
- $\{T, N, B\}$ is an orthonormal basis (or frame).



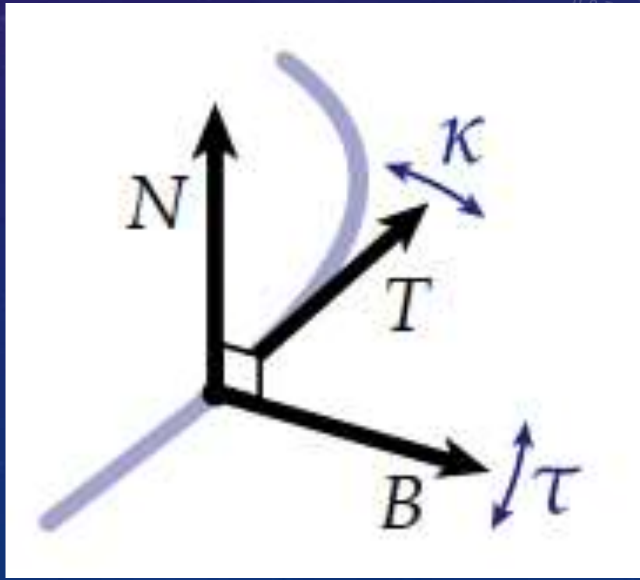
Frenet–Serret frame

- As $\langle T, T \rangle \equiv 1 \Longrightarrow \left\langle \frac{dT}{ds}, T \right\rangle \equiv 0$, set $N = \frac{dT}{ds} / | \frac{dT}{ds} |$ $(\kappa = \left| \frac{dT}{ds} \right| \neq 0)$
- Torsion $\tau : \langle N, N \rangle \equiv 1 \Longrightarrow \left\langle \frac{dN}{ds}, N \right\rangle \equiv 0$, set $\frac{dN}{ds} = \sigma T + \tau B$
- We have $\sigma = \left\langle \frac{dN}{ds}, T \right\rangle = -\left\langle \frac{dT}{ds}, N \right\rangle = -\kappa, \ \frac{dB}{ds} = -\tau N$

Fundamental theorem of space curves

- Given the curvature and torsion of an arc-length parameterized space curve, we can recover the curve itself.

$$\frac{d}{ds}\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix}\begin{bmatrix} T \\ N \\ B \end{bmatrix}$$



More than Frenet–Serret frame

- $\left\langle \frac{dT}{ds}, T \right\rangle \equiv 0, N = \frac{dT}{ds} / \left| \frac{dT}{ds} \right| \Longrightarrow$ select any normal field $\langle N, T \rangle \equiv 0$

$$\frac{d}{ds}\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & \kappa_2 \\ -\kappa_1 & 0 & \tau \\ -\kappa_2 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

- Twist $\tau = \left\langle \frac{dN}{ds}, B \right\rangle = -\left\langle \frac{dB}{ds}, N \right\rangle$

Bishop frame(no twist)

- $\tau \equiv 0$: How to compute? Solving initial value PDE :

$$\begin{cases} \dfrac{dN}{ds} = -\kappa_1 T = -\left\langle \dfrac{dT}{ds}, N \right\rangle T \\ N(0) = N_0 \end{cases}$$

- For a close curve, not always has Bishop frame, e.g. $N(L) \neq N(0)$

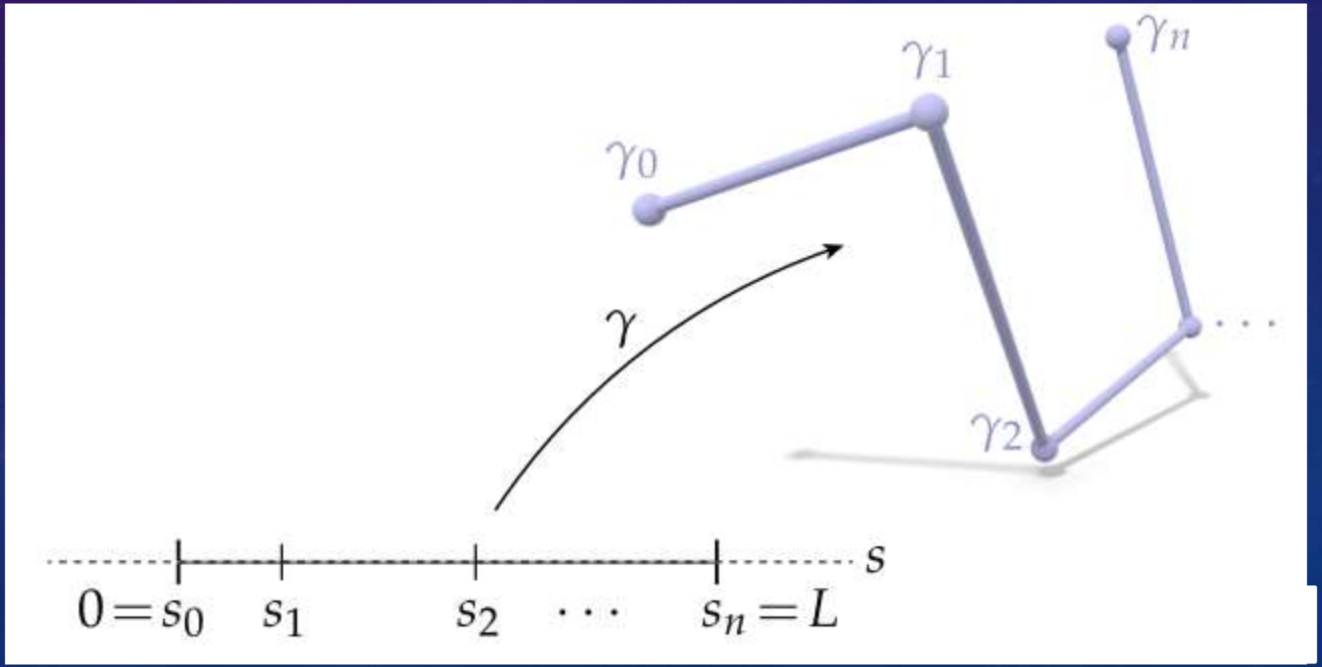
Frame with constant twist

- Definition: $\tau \equiv c$.
- Any frame can be modified into a uniformly twisted frame without changing the total twist $\mathcal{T} = \int_0^L \tau(s)ds$.
- $c = \frac{\mathcal{T}}{L}$, we have $\begin{cases} \frac{dN}{ds} = -\kappa_1 T = -\left\langle \frac{dT}{ds}, N \right\rangle T + \frac{\mathcal{T}}{L}(T \times N) \\ N(0) = N_0 \end{cases}$

Discrete Space Curve

Polygonal curve

- A discrete space curve, or more precisely a polygonal curve is a map $\gamma : I \to \mathbb{R}^3$ where I is the ordered index set $I = (s_0, s_1, \ldots, s_{n-1}, s_n)$



Discretization

- Differential → edge vector: $(d\gamma)_{ij} = \gamma_j - \gamma_i$
- Discrete tangent: $T_{ij} = (d\gamma)_{ij}/|(d\gamma)_{ij}|$.
- A polygonal curve is called regular if $\left|(d\gamma)_{ij}\right| \neq 0$ and $T_{ij} \neq -T_{ji}$ for all edges.
- Discrete normal plane: $T_{ij}^{\perp} = \left\{v \in \mathbb{R}^3, \langle v, T_{ij} \rangle = 0\right\},\ \{N, B\} \in T_{ij}^{\perp}$
- Discrete curvature: $\alpha_i = \cos^{-1}(\langle T_{i-1,i}, T_{i,i+1} \rangle)$

Discretization

- Discrete Frenet–Serret binormal vector on vertex γ_i:

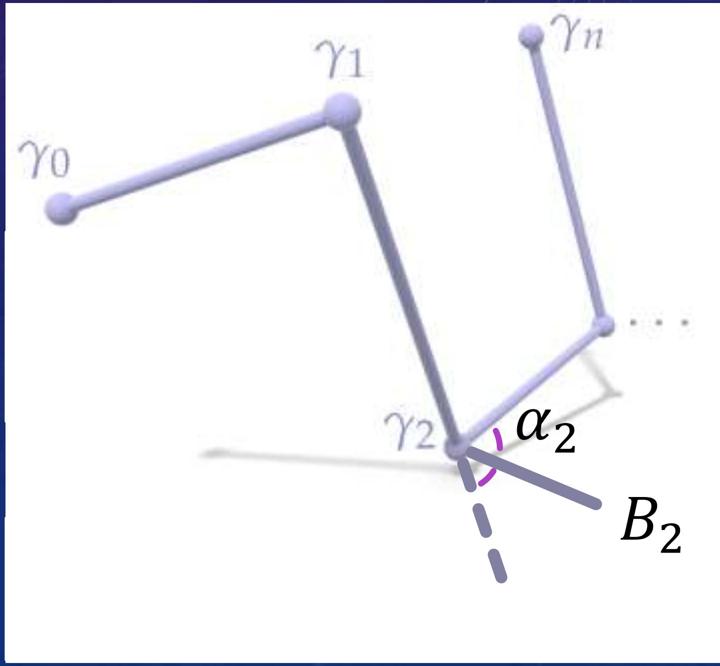
$$B_i = \frac{T_{i-1,i} \times T_{i,i+1}}{|T_{i-1,i} \times T_{i,i+1}|}$$

- Dihedral rotation :

$$R_{B_i}(\alpha_i): T_{i-1,i} \rightarrow T_{i,i+1}$$

- Parallel transport :

$$R_{B_i}(\alpha_i) N_{i-1,i} \rightarrow N_{i,i+1}$$



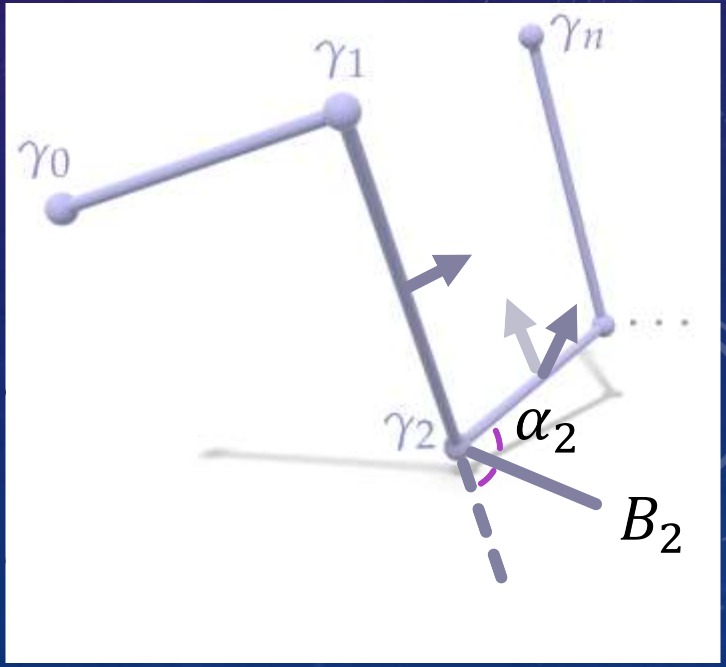
Discretization

- Discrete twist:

$$\beta_i = \cos^{-1}\langle R_{B_i} N_{i-1,i}, N_{i,i+1} \rangle$$

- Constant twist:

$$\beta_i \equiv c$$



Fundamental theorem of discrete space curves

Given:

- Edge lengths $l_{i,i+1}$, curvatures α_i, torsions β_i
- Initial point γ_0, tangent $T_{0,1}$, and normal $N_{0,1}$

For i=1,…n

- $\gamma_i \longleftarrow \gamma_{i-1} + l_{i-1,i} T_{i-1,i}$
- $T_{i,i+1} \longleftarrow R_{T_{i-1,i} \times N_{i-1,i}}(\alpha_i) T_{i-1,i}$
- $N_{i,i+1} \longleftarrow R_{T_{i,i+1}}(\beta_i) R_{T_{i-1,i} \times N_{i-1,i}}(\alpha_i) N_{i-1,i}$

end