Delaunay Triangulations & Voronoi Diagram

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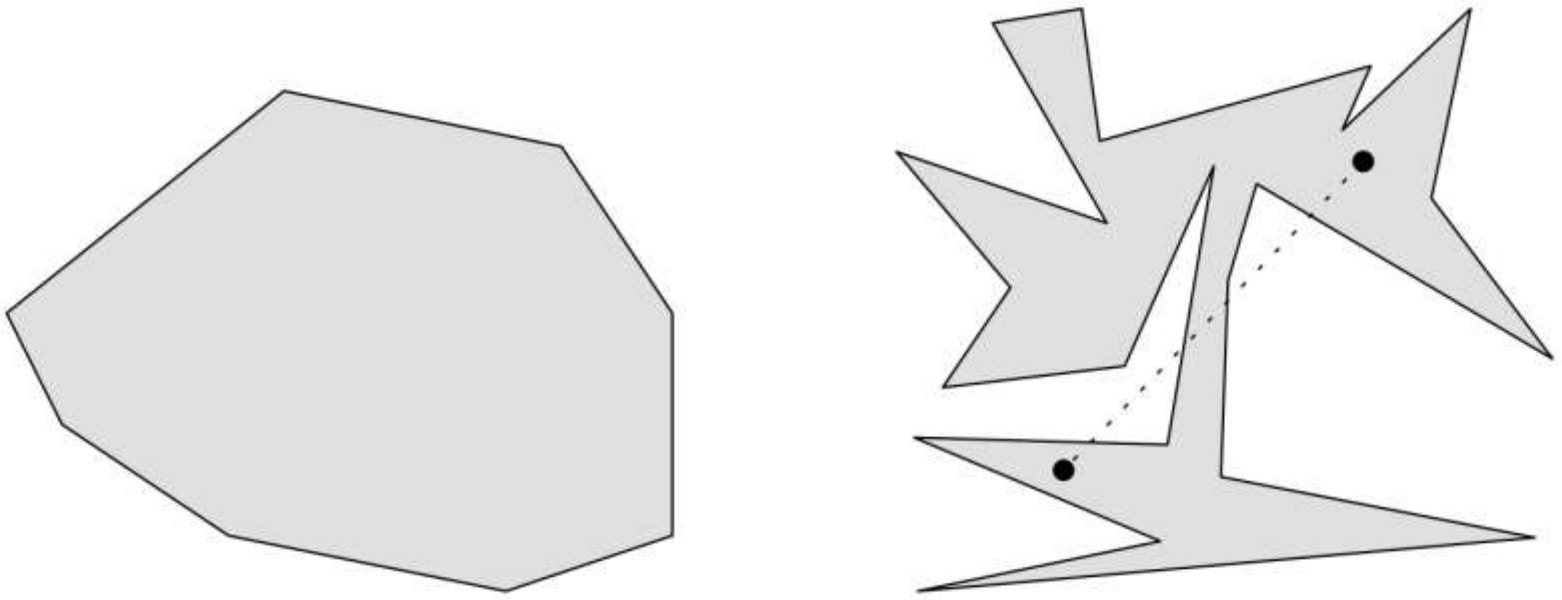
Delaunay Triangulations

Convex polygon

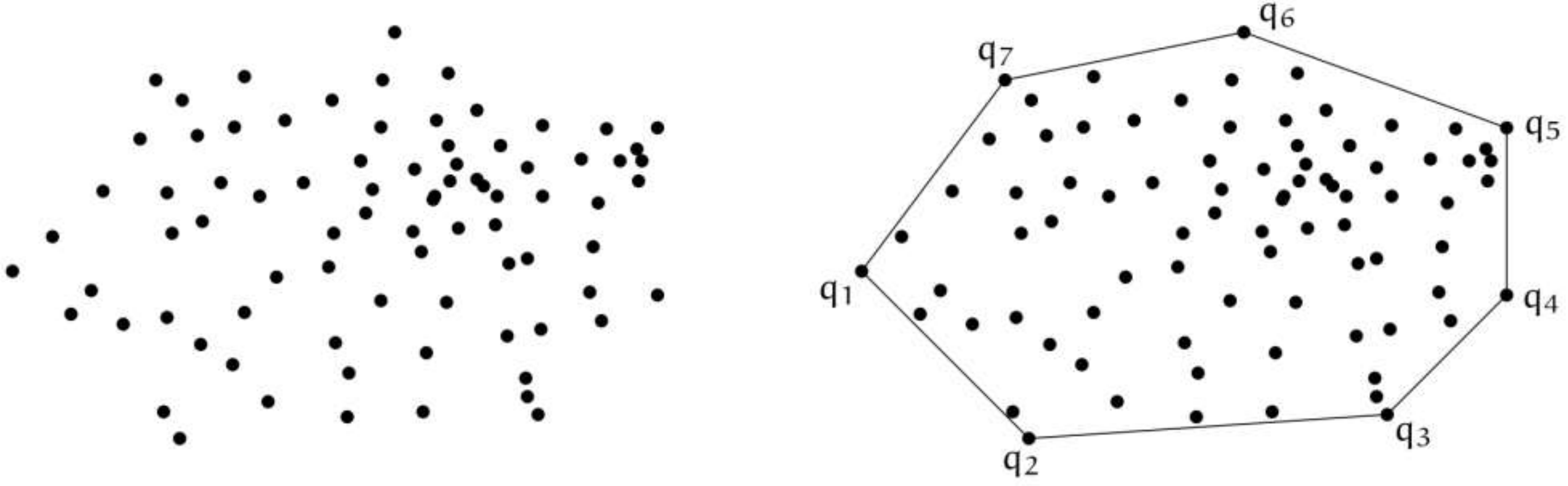
- A set $P \subset \mathbb{R}^2$ is convex if $pq \in P, \forall p, q \in P$.
- For every line $l \in \mathbb{R}^2$, the intersection $l \cap P$ is connected



(a) A convex polygon. (b) A non-convex polygon

Convex hull

- $conv(P)$: convex hull of a finite point set $P \subset \mathbb{R}^2$
- vertex of $conv(P)$: $p \notin conv(P \setminus \{p\})$



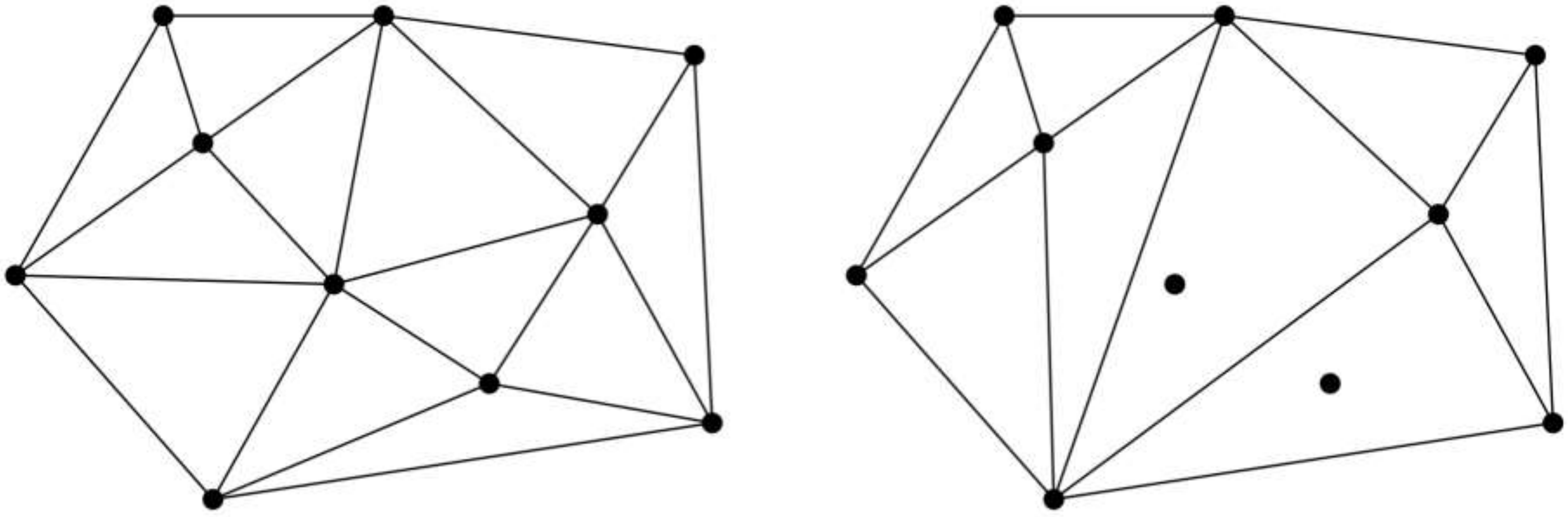
(a) Input. (b) Output.

Trivial algorithms of Convex hull

- Carathéodory's Theorem
  - Test for every point $p \in P$ whether there are $q, r, s \in P \setminus \{p\}$ such that p is inside the triangle with vertices q, r, and s.
  - Runtime $O(n^4)$.
- The Separation Theorem:
  - Test for every pair $(p, q) \in P^2$ whether all points from $P \setminus \{p, q\}$ are to the left of the directed line through p and q (or on the line segment $\overline{pq}$).
  - Runtime $O(n^3)$.

Triangulation of a point set

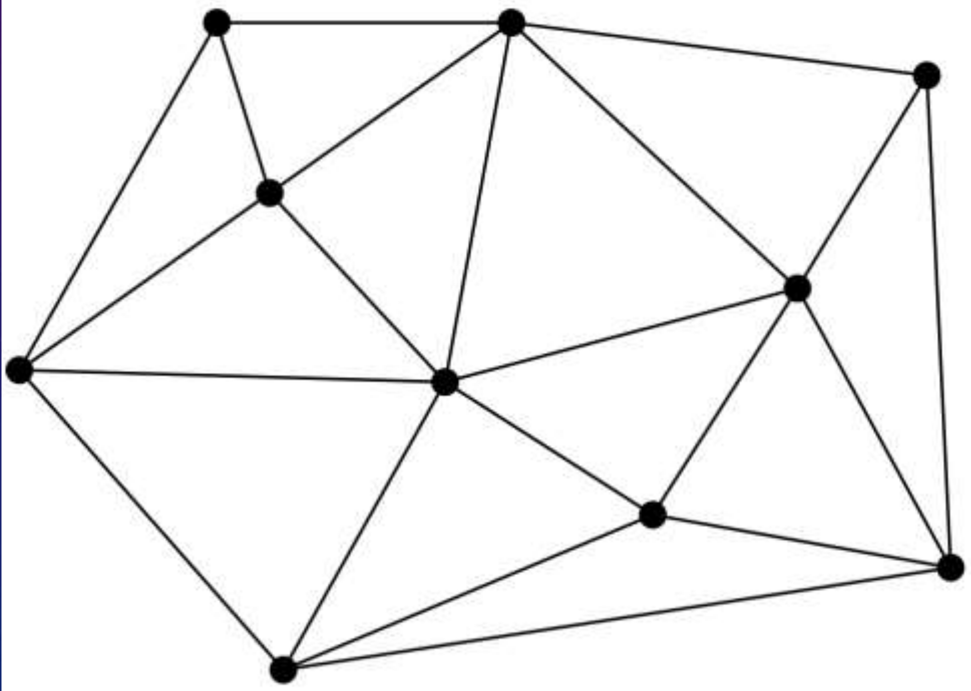
- A triangulation should partition the convex hull while respecting the points in the interior



(b) Point set triangulation. (c) Not a triangulation.

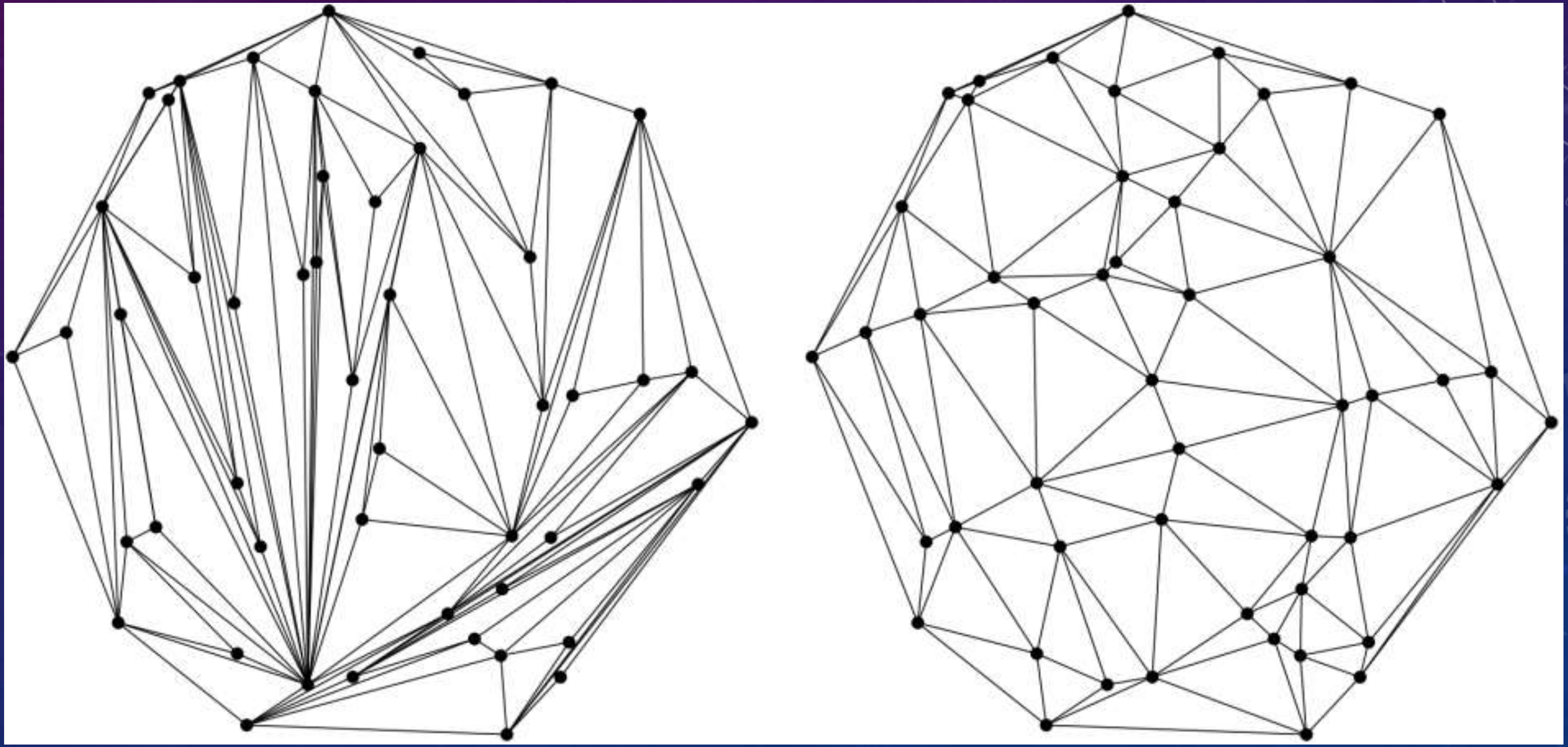
Definition

- A triangulation of a finite point set $P \subset \mathbb{R}^2$ is a collection $\mathcal{T}$ of triangles, such that:
  - $conv(P) = \bigcup_{T \in \mathcal{T}} T$
  - $P = \bigcup_{T \in \mathcal{T}} V(T)$
  - For every distinct pair $S, T \in \mathcal{T}$, the intersection $S \cap T$ is either a common vertex, or a common edge, or empty



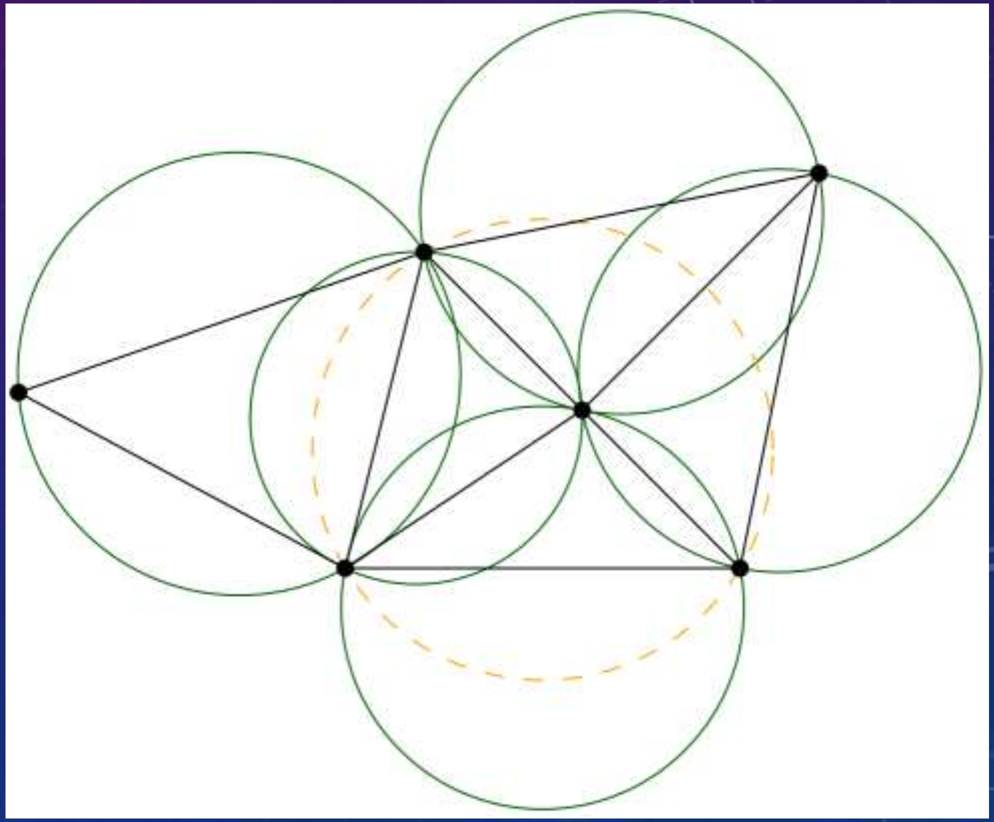
(b) Point set triangulation.

Various triangulations



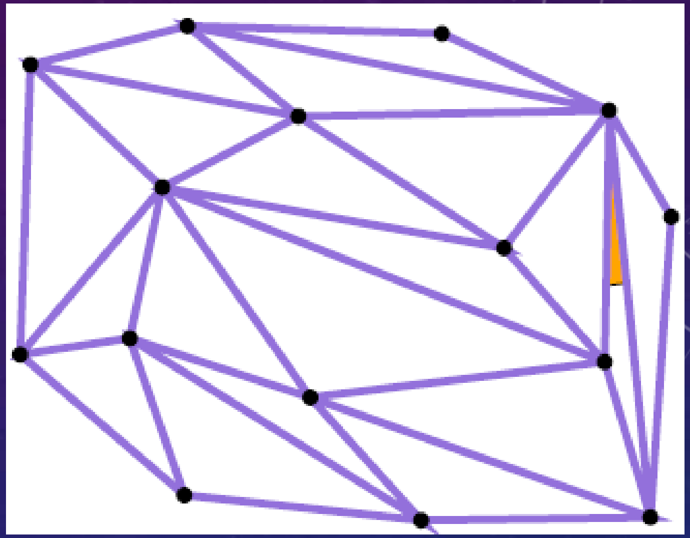
Delaunay triangulation

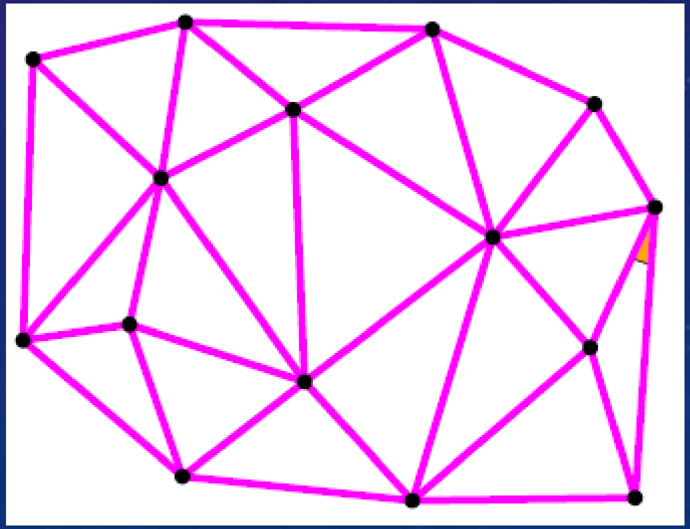
- $DT(P)$: no point in P is inside the circumcircle of any triangle in $DT(P)$.



Delaunay triangulation

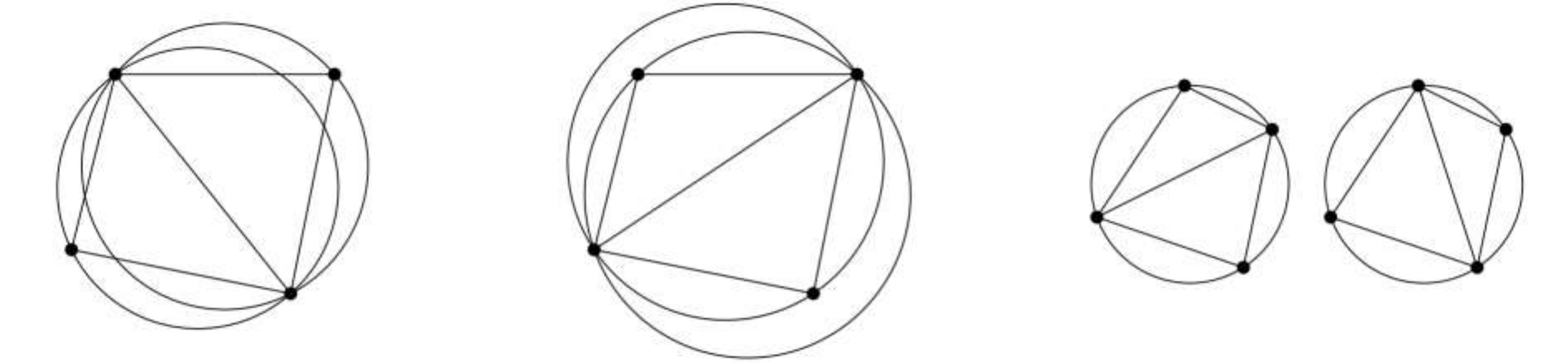
- $DT(P)$: no point in P is inside the circumcircle of any triangle in $DT(P)$.
- DT maximizes the smallest angle.





The Lawson Flip algorithm

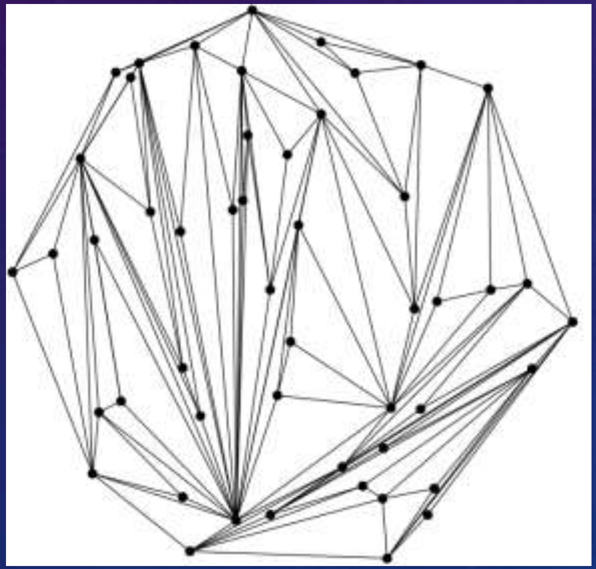
- Edge flip (four points in convex position)

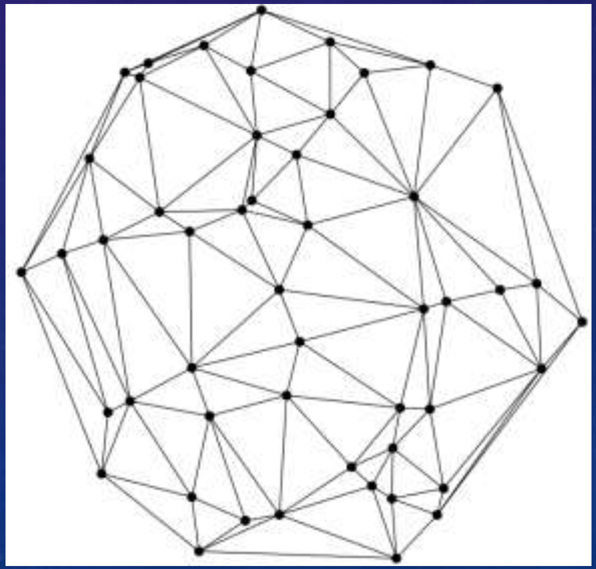


(a) Delaunay triangulation. (b) Non-Delaunay triangulation. (c) Two Delaunay triangulations.

The Lawson Flip algorithm

- Edge flip (four points in convex position)
- Loop in all subtriangulations of four points in convex position.





Theorem

Let $P \subset \mathbb{R}^2$ be a set of n points, equipped with some triangulation $\mathcal{T}$. The Lawson flip algorithm terminates after at most $\binom{n}{2} = O(n^2)$ flips, and the resulting triangulation is a DT of P.

Two-step proof:

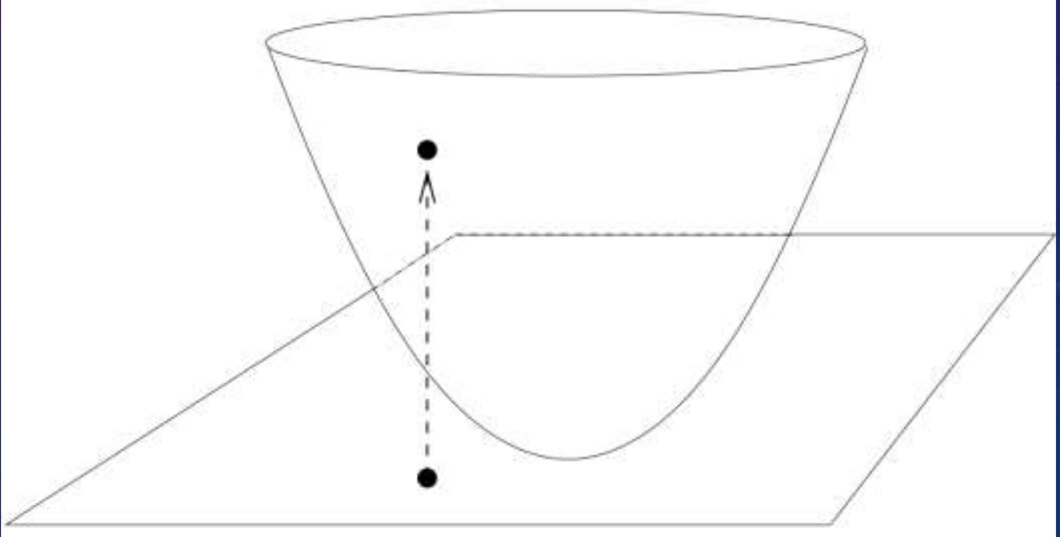
1. The program described above always terminates.
2. The algorithm does what it claims to do, namely the result is a DT.

The Lifting Map

- Given a point $p = (x, y) \in \mathbb{R}^2$, its lifting $l(p)$ is the point

$$l(p) = \left(x, y, x^2 + y^2\right) \in \mathbb{R}^3$$

Unit paraboloid



(a) The lifting map.

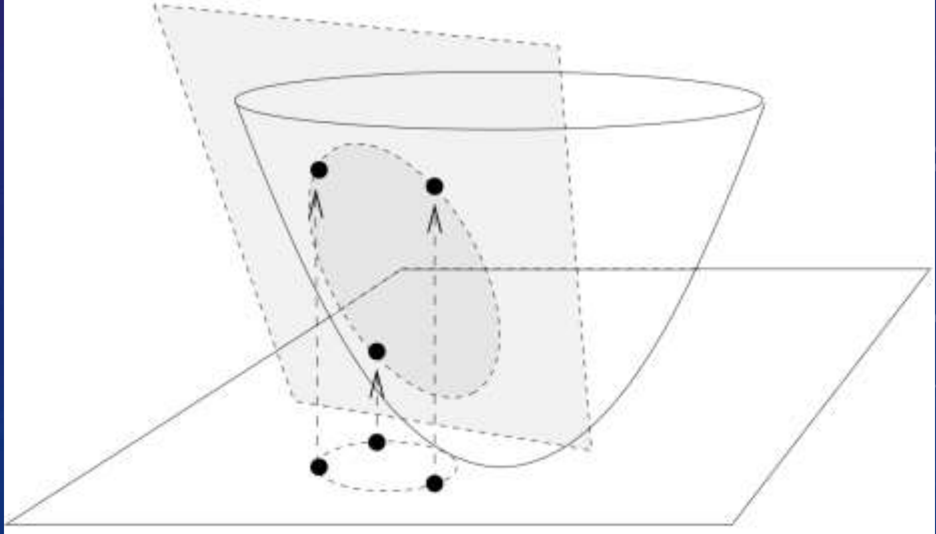
Property

- Lemma: Let $C \subset \mathbb{R}^2$ be a circle of positive radius. The "lifted circle" $l(C) = \{l(p), p \in C\}$ is contained in a unique plane $\mathrm{h}(C) \subset \mathbb{R}^3$.

Proof : $l(p) = \left(x + rcost, y + rsint, x^2 + y^2 + r^2 + 2xrcost + 2yrsint\right)$

Let $q = (x, y, x^2 + y^2 + r^2)$, then

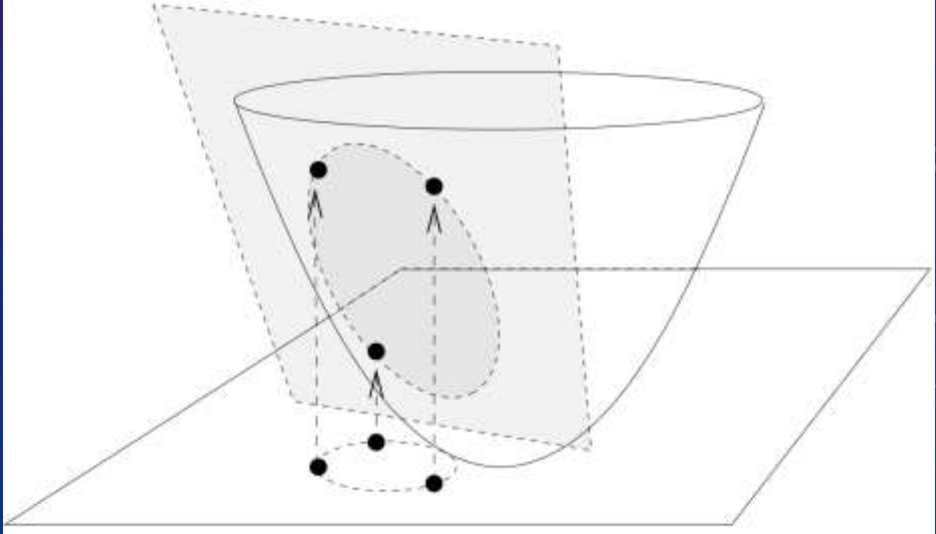
$$\langle l(p) - q, (2x, 2y, -1)\rangle = 0$$



(b) Points on/inside/outside a circle are lifted to points on/below/above a plane.

Property

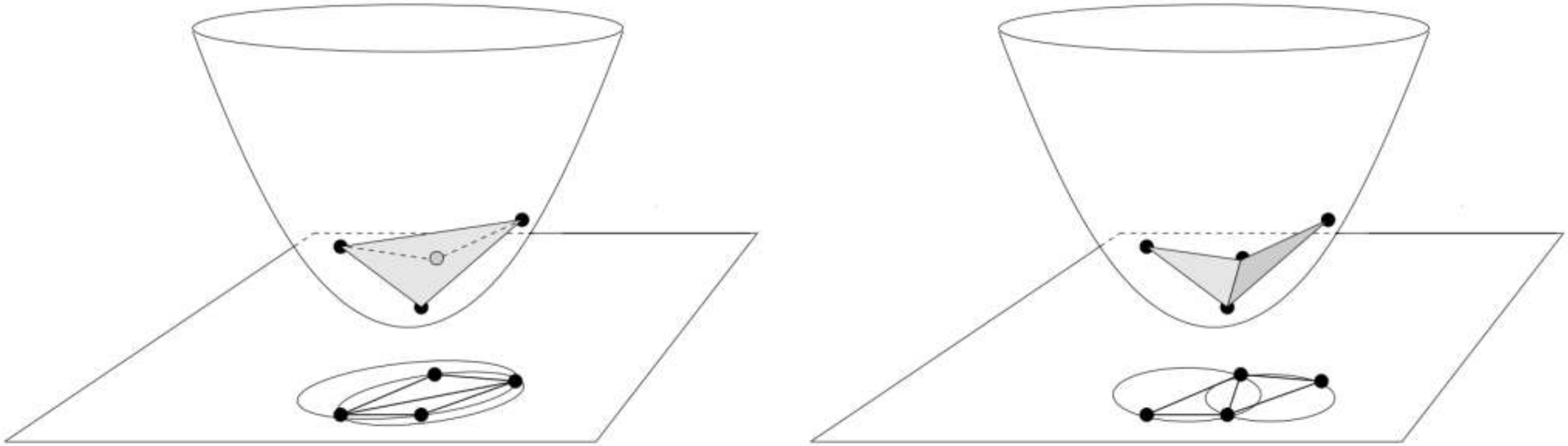
- Lemma: Let $C \subset \mathbb{R}^2$ be a circle of positive radius. The "lifted circle" $l(C) = \{l(p), p \in C\}$ is contained in a unique plane $h(C) \subset \mathbb{R}^3$.
- A point $p \in \mathbb{R}^2$ is strictly inside (outside, respectively) of C if and only if $l(p)$ is strictly below (above, respectively) $h(C)$.



(b) Points on/inside/outside a circle are lifted to points on/below/above a plane.

Termination

- A Lawson flip can therefore be interpreted as an operation that replaces the top two triangles of a tetrahedron by the bottom two ones.

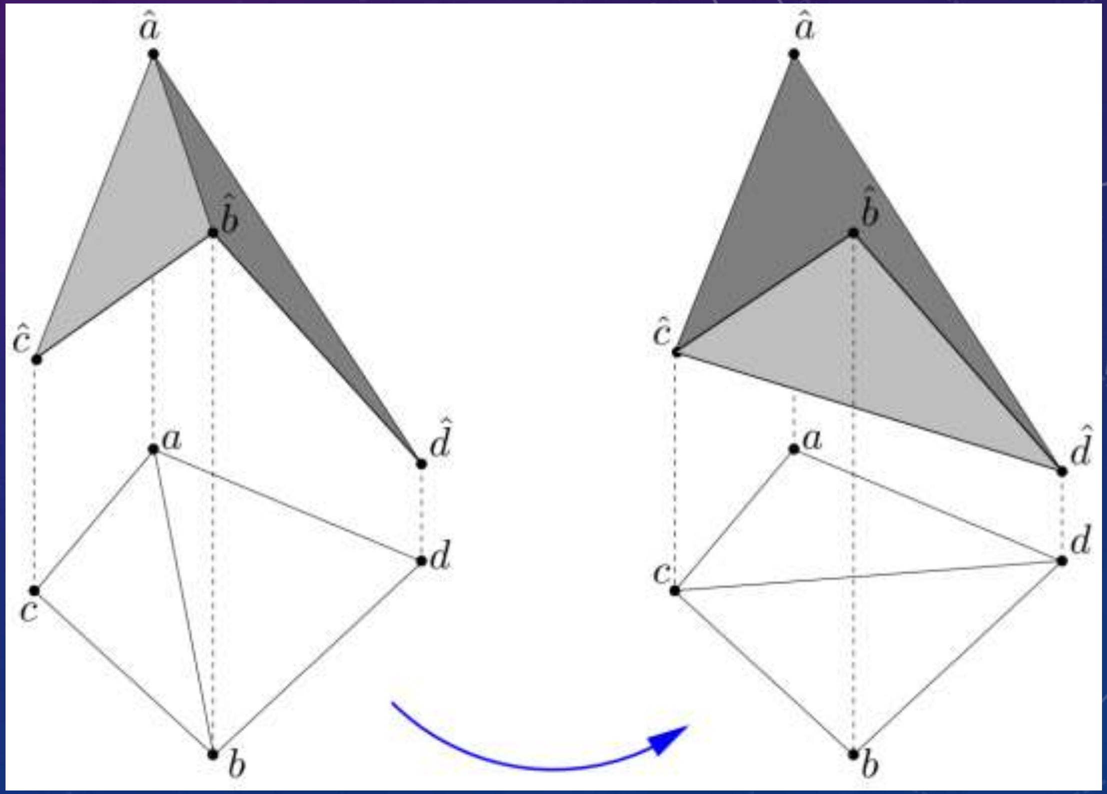


(a) Before the flip: the top two triangles of the tetrahedron and the corresponding non-Delaunay triangulation in the plane.

(b) After the flip: the bottom two triangles of the tetrahedron and the corresponding Delaunay triangulation in the plane.

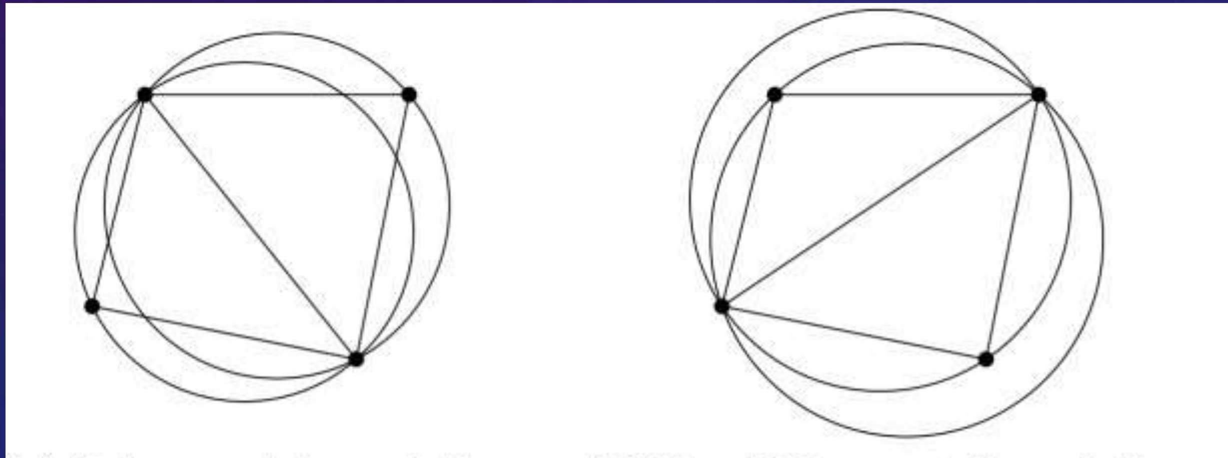
Termination

- Lawson flips decrease the height of the lifted image of triangulation.
- Once an edge has been flipped, it will be strictly above the resulting surface and never be flipped a second time.
- n points span at most $\binom{n}{2}$ edges



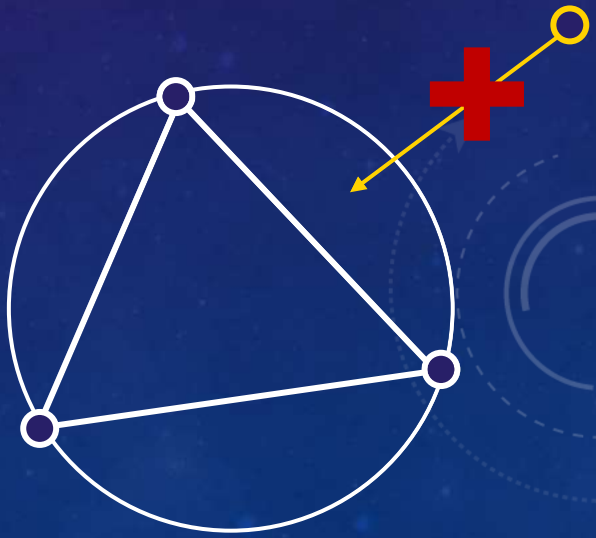
The result is a DT

- Locally Delaunay: Let Δ, Δ′ be two adjacent triangles in the triangulation D that results from the Lawson flip algorithm. Then the circumcircle of Δ does not have any vertex of Δ′ in its interior, and vice versa.



(a) Delaunay triangulation. (b) Non-Delaunay triangulation.

four points in convex position

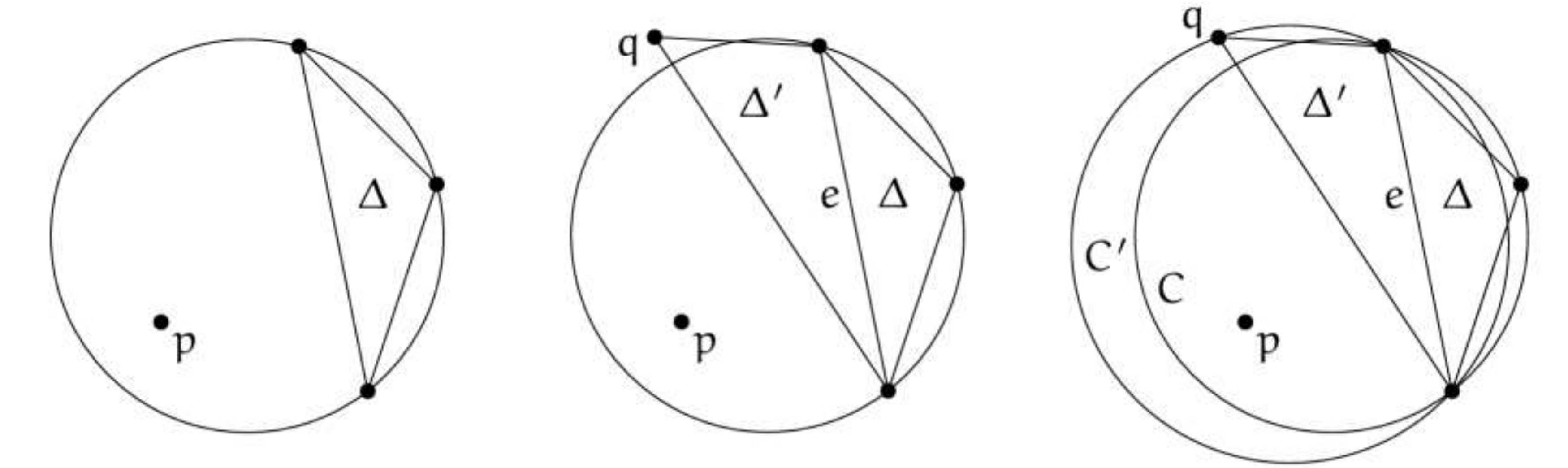


not in convex position

The result is a DT

- Locally Delaunay $\Longleftrightarrow$globally Delaunay

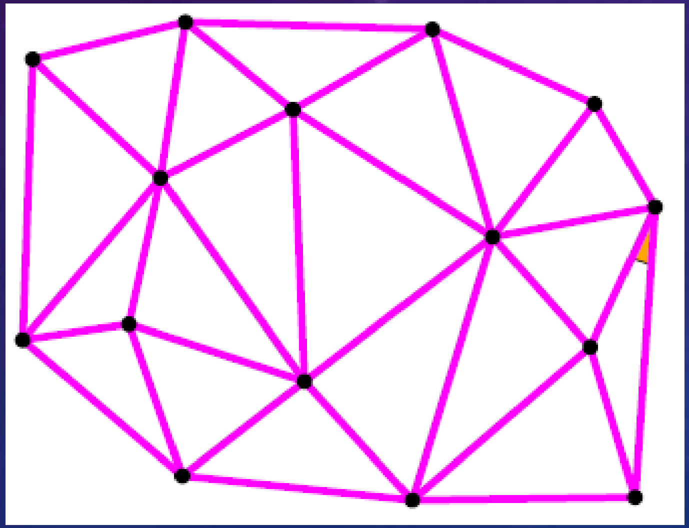
Proof: all piars $\{(\Delta, p), p \in C(\Delta)\}$. Selete the pair with minimum $dist(p, \Delta)$

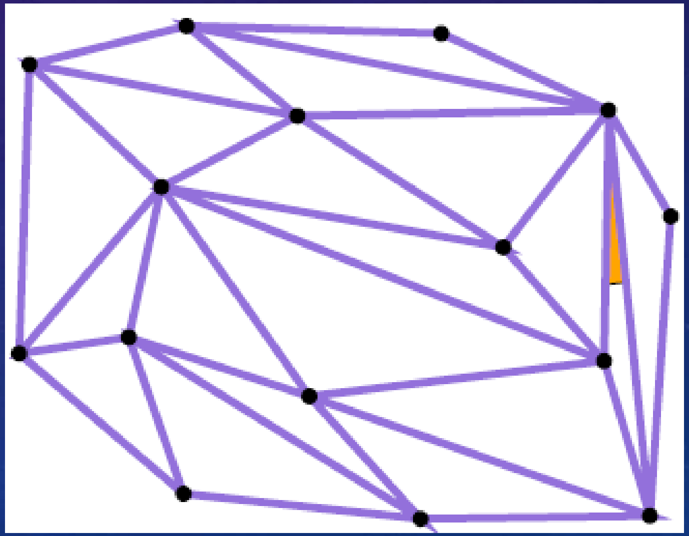


(a) A point p inside the circumcircle C of a triangle Δ. (b) The edge e of Δ closest to p and the second triangle Δ′ incident to e. (c) The circumcircle C′ of Δ′ also contains p, and p is closer to Δ′ than to Δ.

Maximize the minimum angle

- If there is a long and skinny triangle in a Delaunay triangulation, then there is an at least as long and skinny triangle in every triangulation of the point set.





Maximize the minimum angle

- A flip replaces six interior angles by six other interior angles, and we will actually show that the smallest of the six angles strictly increases under the flip.

Before the flip:

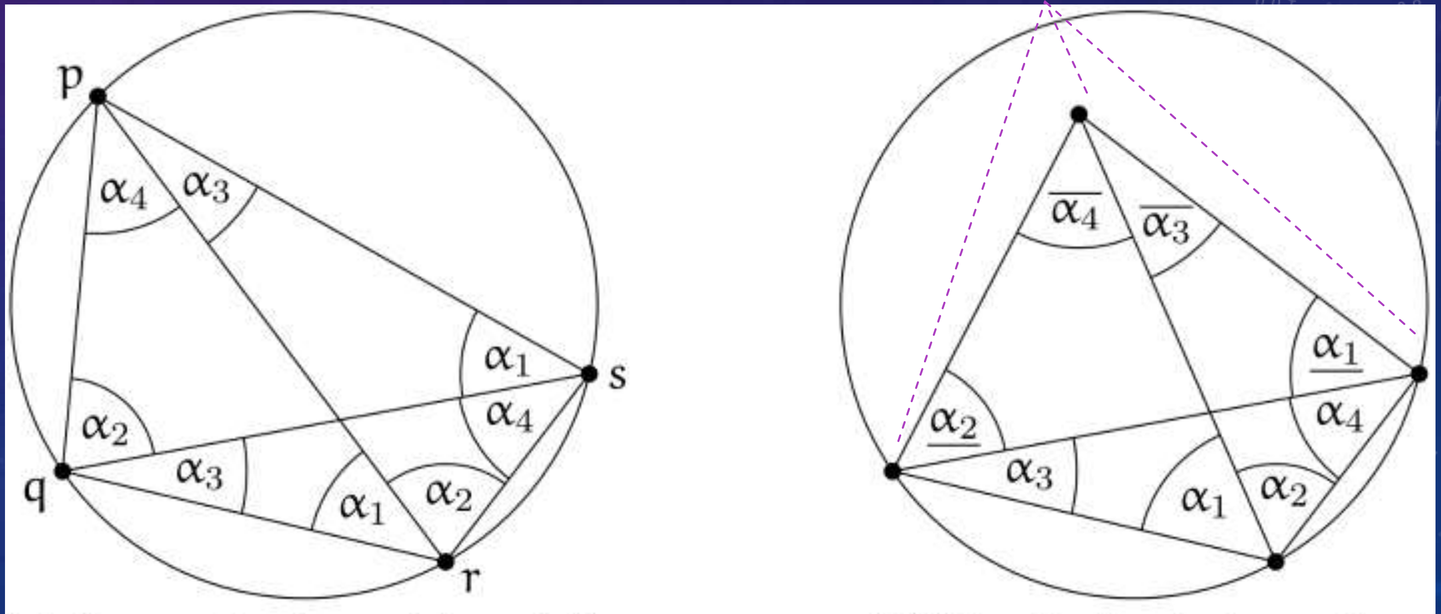
$$\alpha_1 + \alpha_2, \alpha_3, \alpha_4, \underline{\alpha_1}, \underline{\alpha_2}, \overline{\alpha_3} + \overline{\alpha_4}$$

After the flip:

$$\alpha_1, \alpha_2, \overline{\alpha_3}, \overline{\alpha_4}, \underline{\alpha_1} + \alpha_4, \underline{\alpha_2} + \alpha_3$$

$$\alpha_1 > \underline{\alpha_1}, \alpha_2 > \underline{\alpha_2}, \overline{\alpha_3} > \alpha_3, \overline{\alpha_4} > \alpha_4,$$

$$\underline{\alpha_1} + \alpha_4 > \alpha_4, \underline{\alpha_2} + \alpha_3 > \alpha_3$$

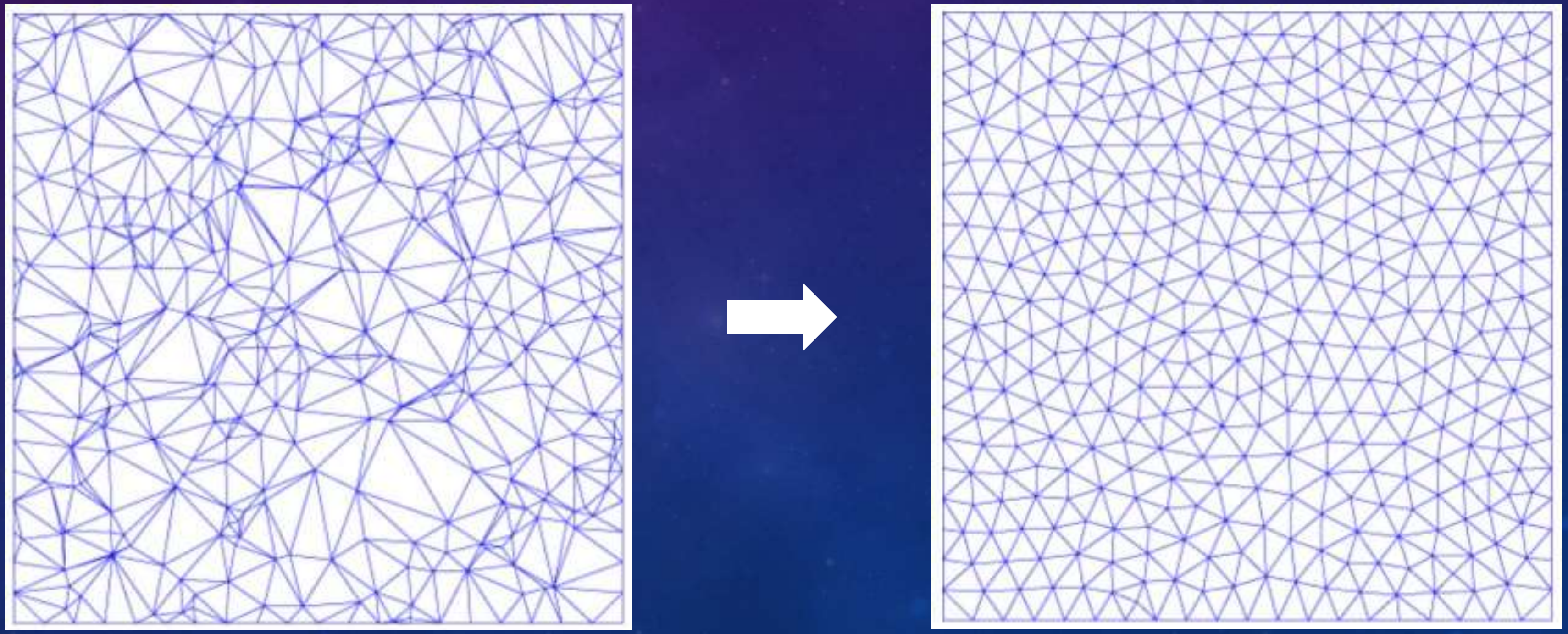


(a) Four cocircular points and the induced eight angles.

(b) The situation before a flip.

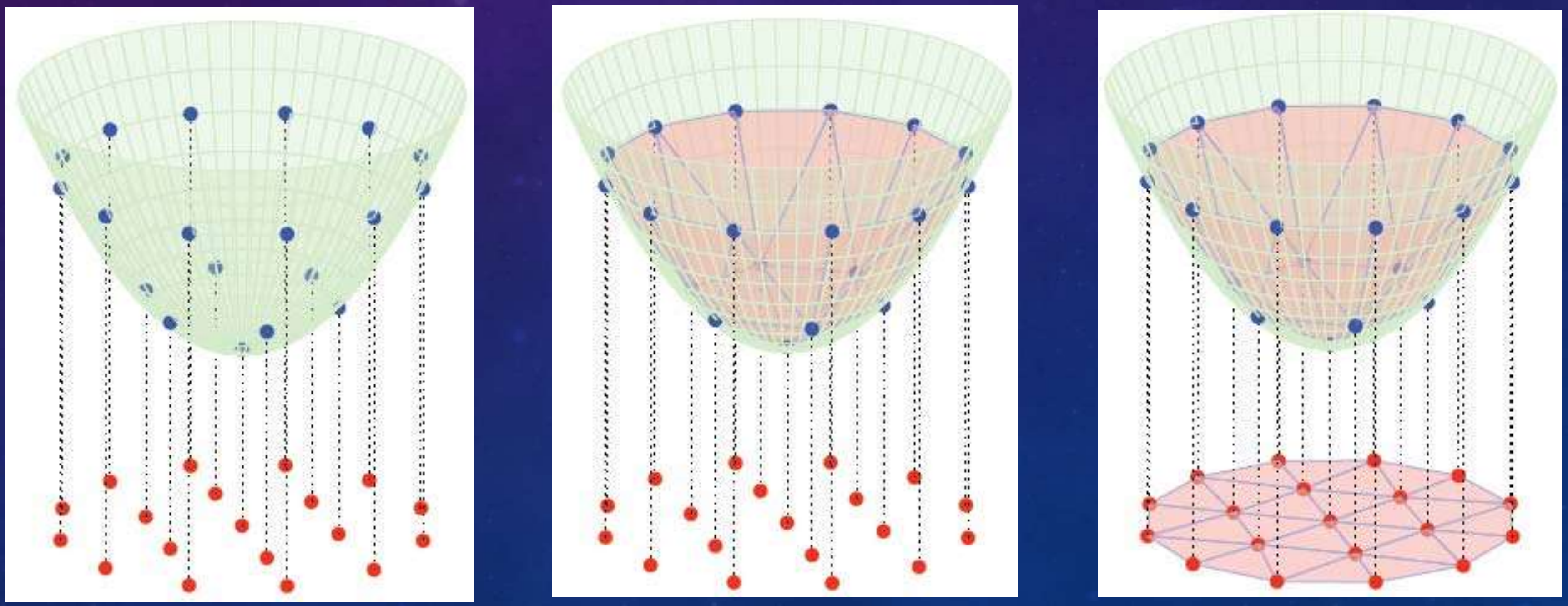
DT is not always a good mesh

- DT only optimize the connectivity as points are fixed.
- Need to optimize vertex positions simultaneously.



Optimal Delaunay triangulation(ODT)

- Fix $P \subset \mathbb{R}^2$, optimize $\mathcal{T}$ s.t. lifted picewise linear image close to unit paraboloid.
- Fix $\mathcal{T}$, optimize $P \subset \mathbb{R}^2$ s.t. lifted picewise linear image close to unit paraboloid.

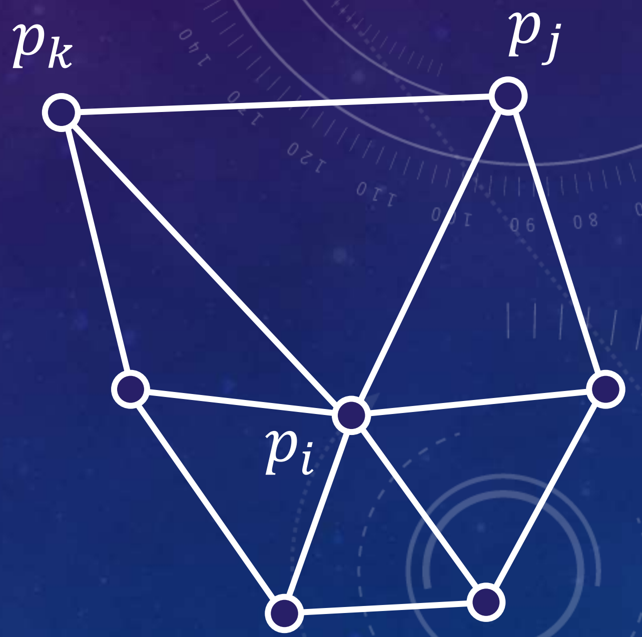


Optimal Delaunay triangulation(ODT)

- $u(p) = x^2 + y^2$ and $\hat{u}(p)$: piecewise linear interpolation
- $E = \sum_{T\in\mathcal{T}} \int_T |\hat{u}(p) - u(p)|dp = \sum_{T\in\mathcal{T}} \int_T \hat{u}(p)dp + C$

$$= \sum_{T_{ijk}\in\mathcal{T}} \frac{|T_{ijk}|}{3}(u(p_i) + u(p_j) + u(p_k)) + C$$

Fix all other points except p_i:

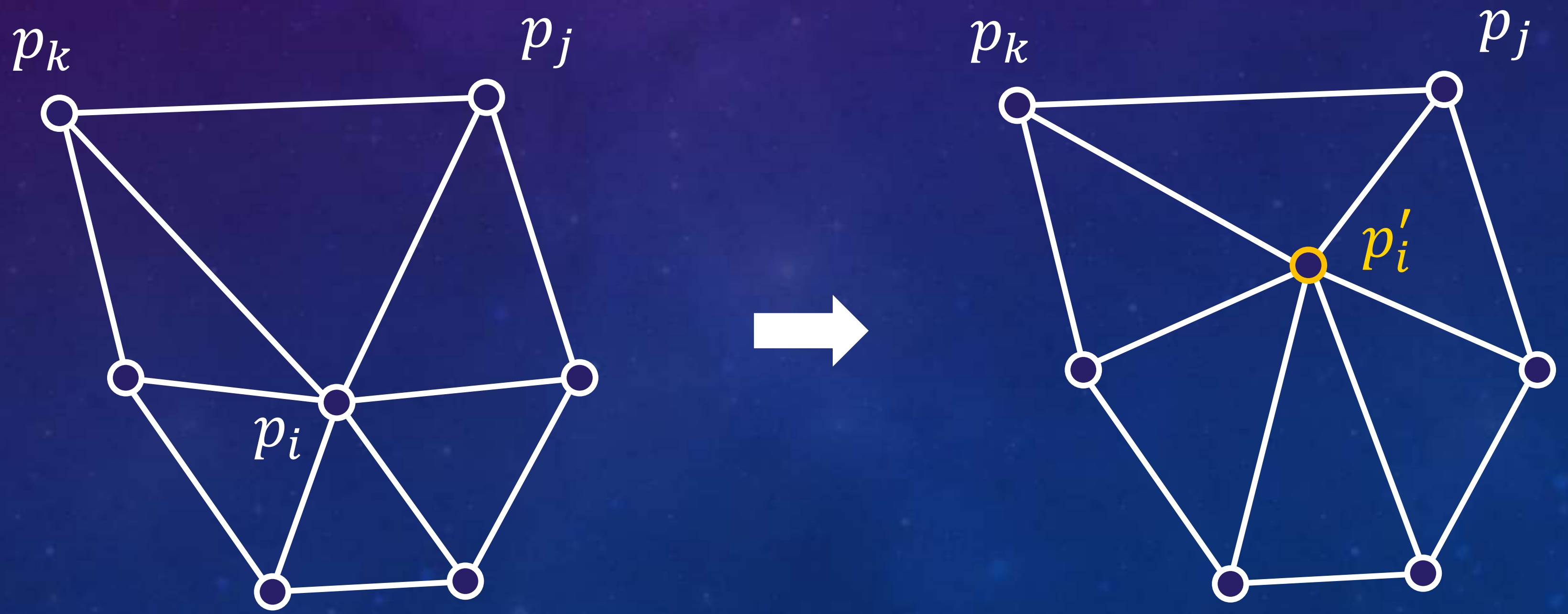


$$\nabla_{p_i} E = \sum_{T_{ijk}\in\Omega_i} \frac{\nabla_{p_i}|T_{ijk}|}{3}(u(p_i) + u(p_j) + u(p_k)) + \sum_{T_{ijk}\in\Omega_i} \frac{|T_{ijk}|}{3}\nabla_{p_i} u(p_i) = 0$$

Property of $\nabla_{p_i}|T_{ijk}|$

- Property 1. $\sum_{T_{ijk}\in\Omega_i} \nabla_{p_i}|T_{ijk}| = 0$

Proof: $\sum_{T_{ijk}\in\Omega_i} \nabla_{p_i}|T_{ijk}| = \nabla_{p_i} \sum_{T_{ijk}\in\Omega_i}|T_{ijk}| = \nabla_{p_i} C = 0$



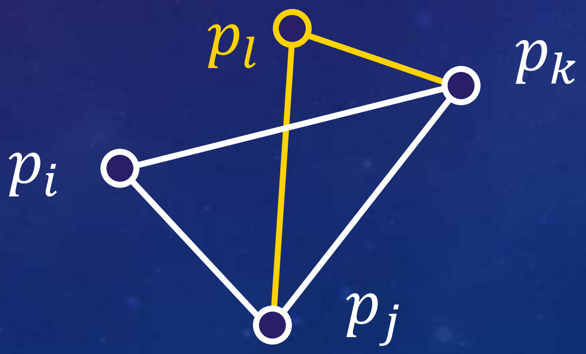
Property of $\nabla_{p_i}|T_{ijk}|$

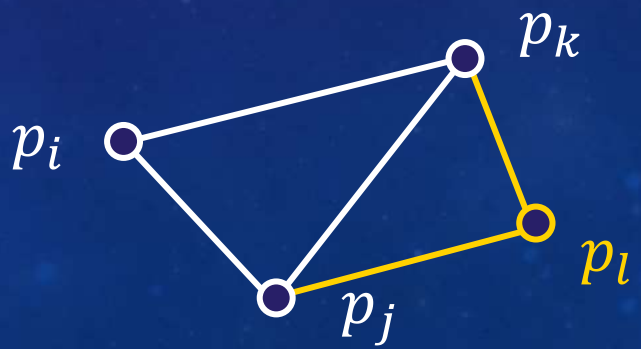
- Property 2. for p_i, p_l in one side of $\overline{p_j p_k}$, $\nabla_{p_i}|T_{ijk}| = \nabla_{p_l}|T_{ljk}|$

  for p_i, p_l in different sides of $\overline{p_j p_k}$, $\nabla_{p_i}|T_{ijk}| = -\nabla_{p_l}|T_{lkj}|$

Proof: denote $r(v) = e^{\frac{\pi}{2}i}v$ represents the 90° degree anticlockwise rotation.

Then, $\nabla_{p_i}|T_{ijk}| = \frac{1}{2}r(p_k - p_j), \nabla_{p_l}|T_{lkj}| = \frac{1}{2}r(p_j - p_k)$.





Property of $\nabla_{p_i}|T_{ijk}|$

- Property 3. for the triangle T_{ijk}, $\nabla_{p_i}|T_{ijk}| + \nabla_{p_j}|T_{ijk}| + \nabla_{p_k}|T_{ijk}| = 0$

Proof: $\nabla_{p_i}|T_{ijk}| + \nabla_{p_j}|T_{ijk}| + \nabla_{p_k}|T_{ijk}|$

$$= \frac{1}{2}r(p_k - p_j) + \frac{1}{2}r(p_i - p_k) + \frac{1}{2}r(p_j - p_i) = 0.$$

Optimal Delaunay triangulation(ODT)

Fix all other points except p_i:

$$\nabla_{p_i} E = \sum_{T_{ijk} \in \Omega_i} \frac{\nabla_{p_i} |T_{ijk}|}{3} (u(p_i) + u(p_j) + u(p_k)) + \sum_{T_{ijk} \in \Omega_i} \frac{|T_{ijk}|}{3} \nabla_{p_i} u(p_i) = 0$$

As $\nabla_{p_i} u(p_i) = \nabla_{p_i} \|p_i\|^2 = 2p_i$, denote $|\Omega_i| \triangleq \sum_{T_{ijk} \in \Omega_i} \frac{|T_{ijk}|}{3}$,

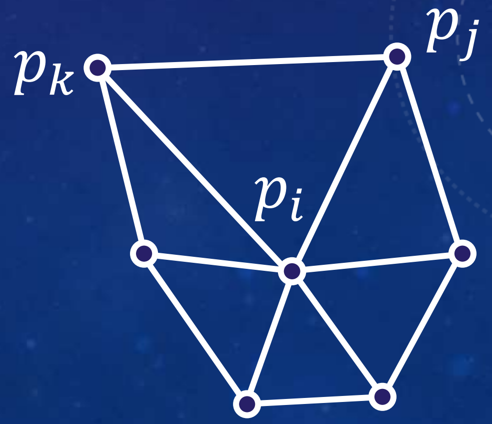
$$p_i^* = -\sum_{T_{ijk} \in \Omega_i} \frac{\nabla_{p_i} |T_{ijk}|}{6|\Omega_i|} \left(u(p_j) + u(p_k)\right) = \sum_{T_{ijk} \in \Omega_i} \frac{\nabla_{p_j} |T_{ijk}| + \nabla_{p_k} |T_{ijk}|}{6|\Omega_i|} \left(u(p_j) + u(p_k)\right)$$

Optimal Delaunay triangulation(ODT)

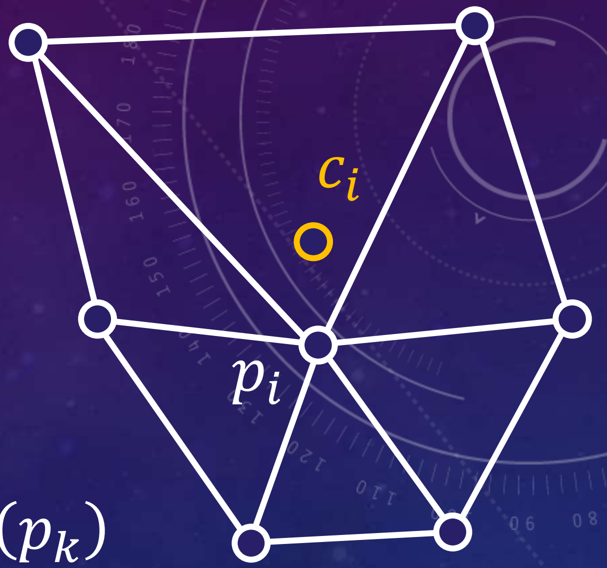
$$p_i^* = -\sum_{T_{ijk}\in\Omega_i} \frac{\nabla_{p_i}|T_{ijk}|}{6|\Omega_i|}\Big(u(p_j)+u(p_k)\Big) = \sum_{T_{ijk}\in\Omega_i} \frac{\nabla_{p_j}|T_{ijk}|+\nabla_{p_k}|T_{ijk}|}{6|\Omega_i|}\Big(u(p_j)+u(p_k)\Big)$$

$$p_i^* = \frac{1}{6|\Omega_i|}\sum_{T_{ijk}\in\Omega_i} \nabla_{p_j}|T_{ijk}|u(p_j) + \nabla_{p_j}|T_{ijk}|u(p_k) + \nabla_{p_k}|T_{ijk}|u(p_j) + \nabla_{p_k}|T_{ijk}|u(p_k)$$

$$= \frac{1}{6|\Omega_i|}\sum\nolimits_{T_{ijk}\in\Omega_i} \nabla_{p_j}|T_{ijk}|u(p_j) + \nabla_{p_k}|T_{ijk}|u(p_k)$$



Optimal Delaunay triangulation(ODT)



We prove p_i^* is the barycenter c_i of Ω_i

$$p_i^* = \frac{1}{6|\Omega_i|} \sum_{T_{ijk} \in \Omega_i} \nabla_{p_j} |T_{ijk}| u(p_j) + \nabla_{p_k} |T_{ijk}| u(p_k)$$

Lemma: for $\forall q \in \mathbb{R}^2$, let $v(p) \triangleq ||p - q||^2 = u(p) + ||q||^2 - 2q^T p$,

for each T, $\hat{v}(p) - v(p) = \hat{u}(p) - u(p)$. Then

$$E = \sum_{T \in \mathcal{T}} \int_T |\hat{v}(p) - v(p)| dp = \sum_{T \in \mathcal{T}} \int_T \hat{v}(p) dp + C$$

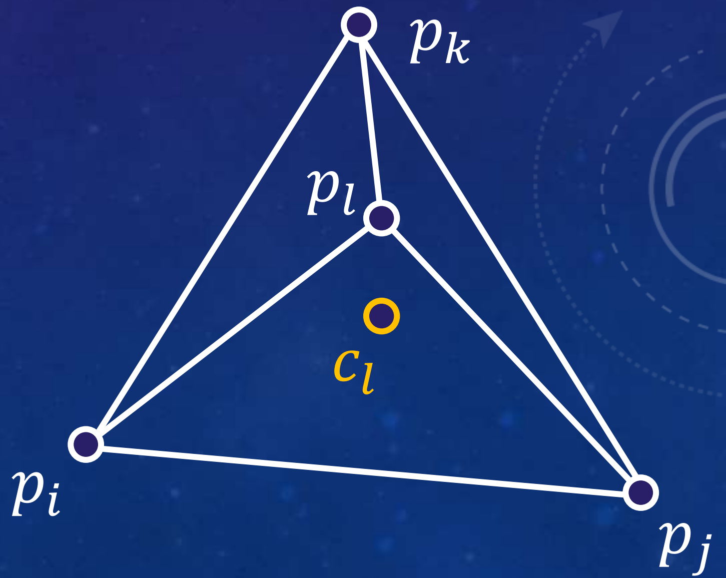
Optimal Delaunay triangulation(ODT)

Fix all other points except p_i, $\nabla_{p_i} v(p_i) = 2(p_i - q)$, similarly

$$p_i^* = q + \frac{1}{6|\Omega_i|} \sum_{T_{ijk} \in \Omega_i} \nabla_{p_j} |T_{ijk}| \|p_j - q\|^2 + \nabla_{p_k} |T_{ijk}| \|p_k - q\|^2$$

We consider a special Ω_l, let $q = c_l$.

Due to $\sum_{T_{ljk} \in \Omega_l} \nabla_{p_l} |T_{ljk}| = 0$, then $p_l^* = c_l$.



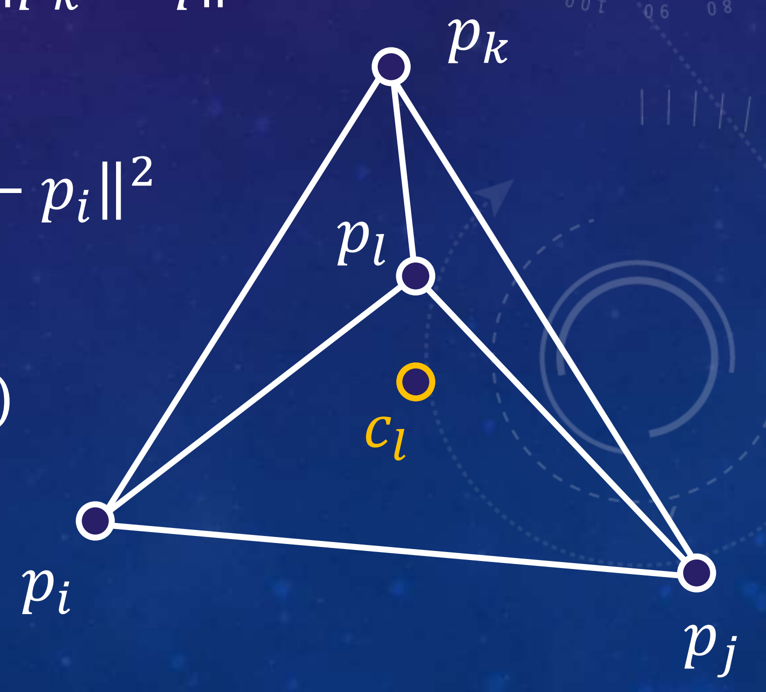
Optimal Delaunay triangulation(ODT)

Let $q = p_i$, then

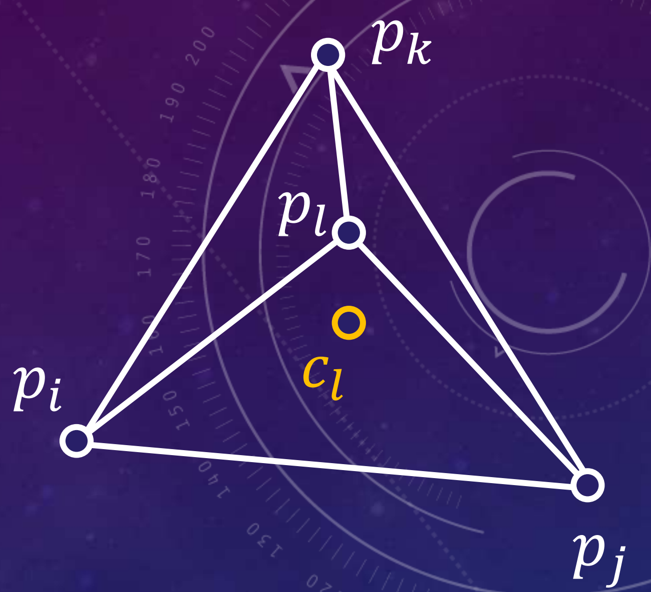
$$c_l = q + \frac{1}{6|\Omega_l|} \sum_{T_{ljk}\in\Omega_i} \nabla_{p_j}|T_{ljk}|\|p_j - q\|^2 + \nabla_{p_k}|T_{ljk}|\|p_k - q\|^2$$

$$\Rightarrow c_l = p_i + \frac{1}{6|\Omega_l|}\Big(\nabla_{p_j}|T_{ljk}|\|p_j - p_i\|^2 + \nabla_{p_k}|T_{ljk}|\|p_k - p_i\|^2$$

$$+\nabla_{p_k}|T_{lki}|\|p_k - p_i\|^2 + \nabla_{p_j}|T_{lij}|\|p_j - p_i\|^2 \Big)$$



Optimal Delaunay triangulation(ODT)



$$c_l = p_i + \frac{1}{6|\Omega_l|}\Big(\nabla_{p_j}|T_{ljk}|\|p_j - p_i\|^2 + \nabla_{p_k}|T_{ljk}|\|p_k - p_i\|^2$$

$$+\nabla_{p_k}|T_{lki}|\|p_k - p_i\|^2 + \nabla_{p_j}|T_{lij}|\|p_j - p_i\|^2 \Big)$$

$$c_l = p_i + \frac{1}{6|\Omega_l|}\Big((\nabla_{p_k}|T_{ljk}| + \nabla_{p_k}|T_{lki}|)\|p_k - p_i\|^2 + (\nabla_{p_j}|T_{ljk}| + \nabla_{p_j}|T_{lij}|)\|p_j - p_i\|^2 \Big)$$

$$= p_i + \frac{1}{2|T_{ijk}|}\left(\nabla_{p_k}|T_{ijk}|\|p_k - p_i\|^2 + \nabla_{p_j}|T_{ijk}|\|p_j - p_i\|^2\right)$$

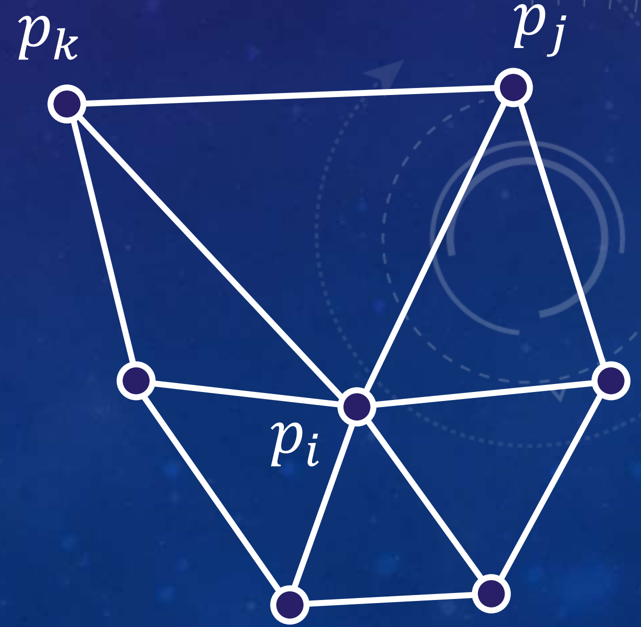
$$\Longrightarrow \nabla_{p_k}|T_{ijk}|\|p_k - p_i\|^2 + \nabla_{p_j}|T_{ijk}|\|p_j - p_i\|^2 = 2|T_{ijk}|(c_l - p_i)$$

Optimal Delaunay triangulation(ODT)

General case: $p_i^* = q + \frac{1}{6|\Omega_i|}\sum_{T_{ijk}\in\Omega_i} \nabla_{p_j}|T_{ijk}|\|p_j - q\|^2 + \nabla_{p_k}|T_{ijk}|\|p_k - q\|^2$

Let $q = p_i$, then $p_i^* = p_i + \frac{1}{6|\Omega_i|}\sum_{T_{ijk}\in\Omega_i} \nabla_{p_j}|T_{ijk}|\|p_j - p_i\|^2 + \nabla_{p_k}|T_{ijk}|\|p_k - p_i\|^2$

Then $p_i^* = p_i + \frac{1}{6|\Omega_i|}\sum_{T_{ijk}\in\Omega_i} 2|T_{ijk}|(c_{ijk} - p_i) = c_i$



Basic algorithm

- Fix $P \subset \mathbb{R}^2$, optimize $\mathcal{T}$ s.t. lifted picewise linear image close to unit paraboloid.
- Fix $\mathcal{T}$, optimize $P \subset \mathbb{R}^2$ s.t. lifted picewise linear image close to unit paraboloid.

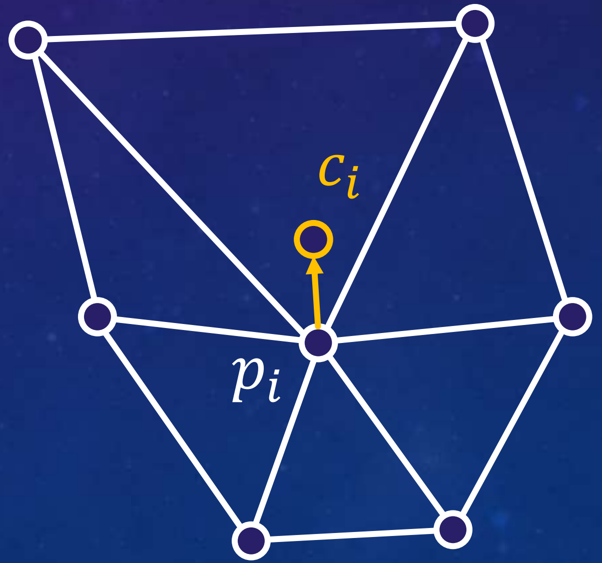
For iter = 1 , … , maxIter

For vertex id i = 1 , … , n

$P_i \leftarrow$ barycenter c_i of Ω_i

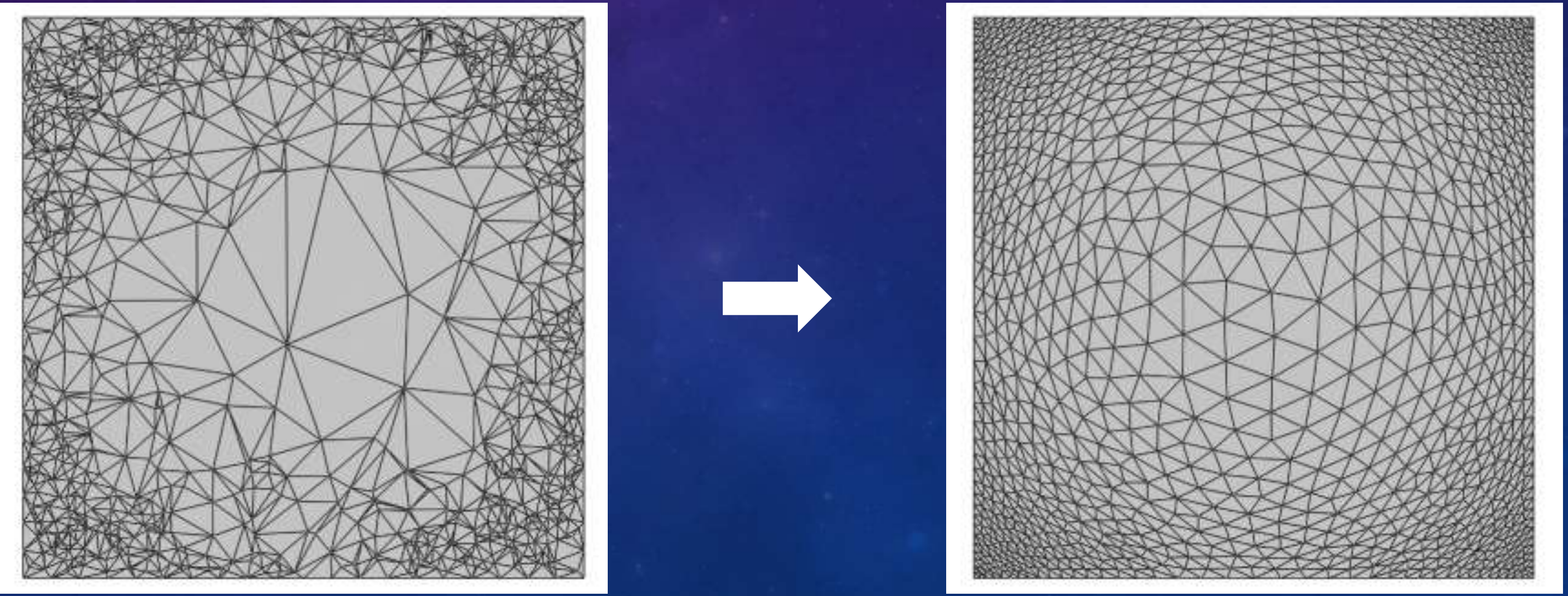
End

End



Generalization

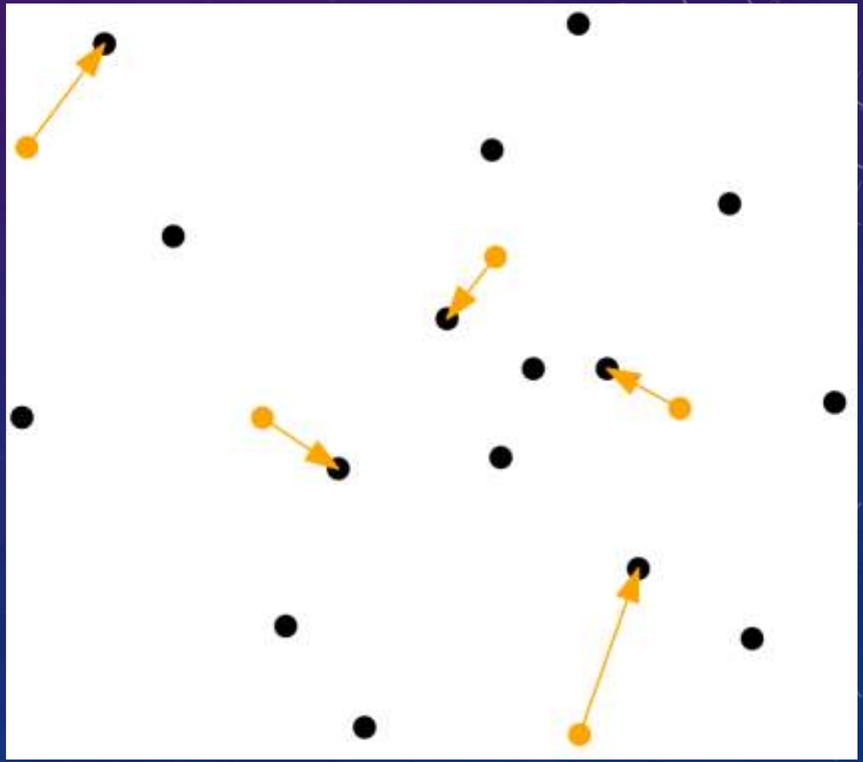
- Non uniform density: $E = \sum_{T \in \mathcal{T}} \int_T |\hat{u}(x) - u(x)| \rho(x) dx$
- Any convex function$u$, i.e. $u(x, y) = e^{\frac{(x^2+y^2)}{10}}, \Omega = [-5,5]^2$



Voronoi Diagram

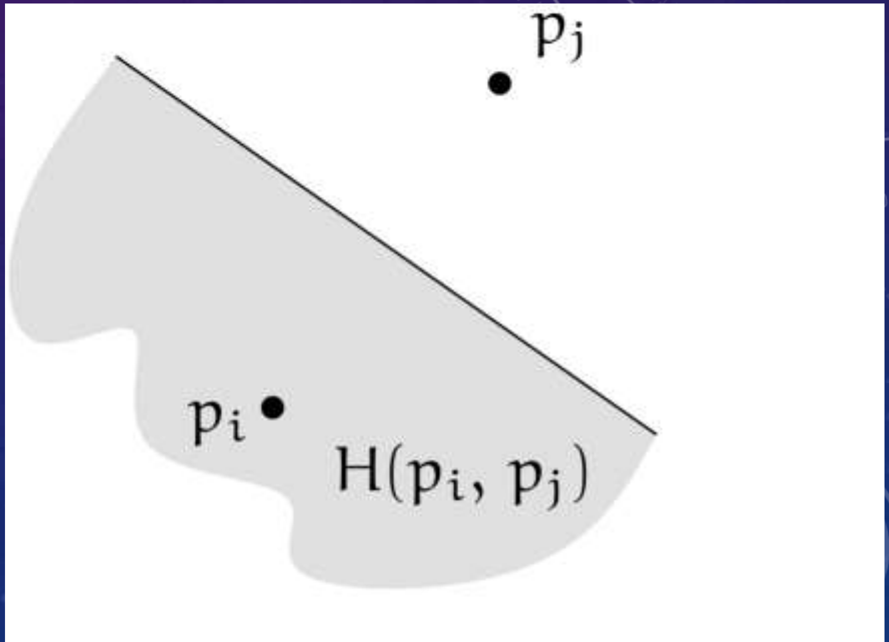
Post Office Problem

- Suppose there are n post offices $p_1, ..., p_n$ in a city.
- Someone who is located at a position q within the city would like to know which post office is closest to him.



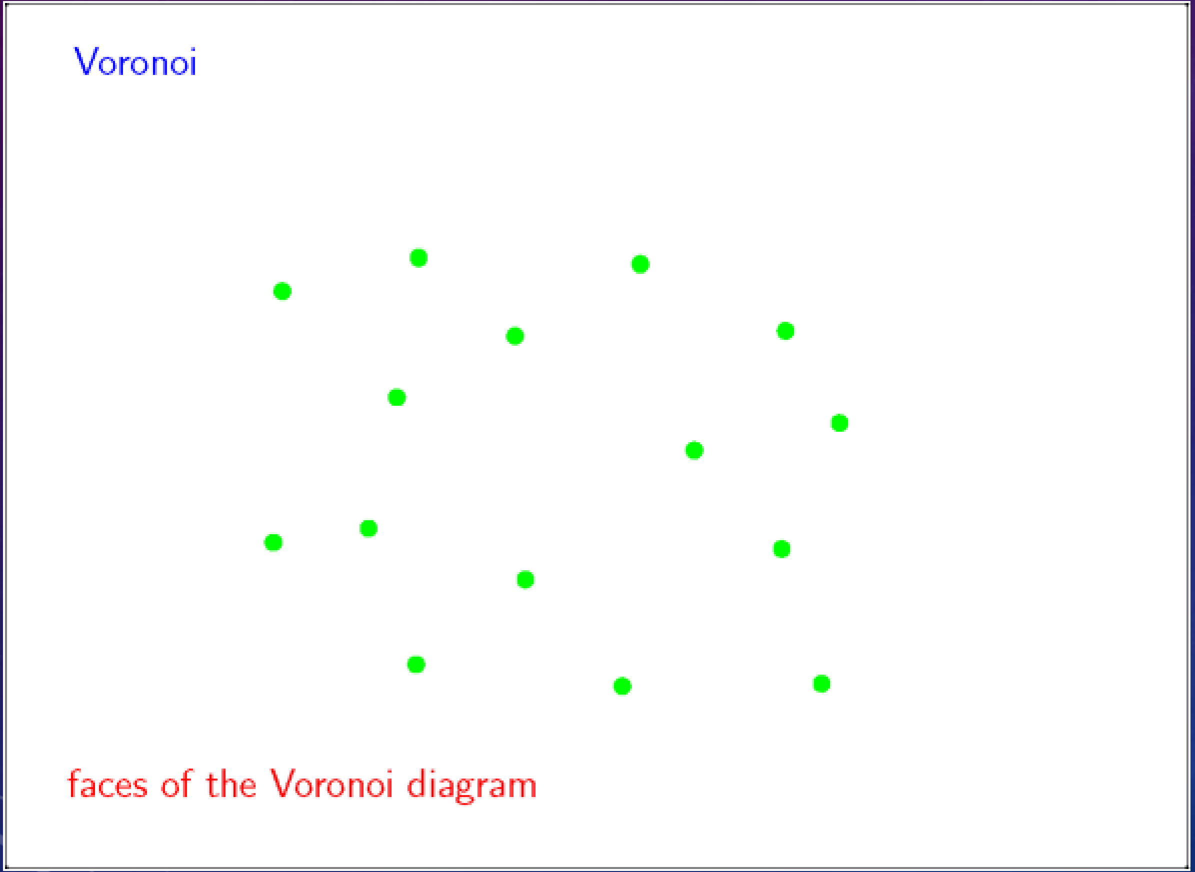
Post Office Problem

- Query in loops (low efficiency)
- Basic idea:
  - Partition the query space into regions on which is the answer is the same.
  - In our case, this amounts to partition the plane into regions such that for all points within a region the same point from P is closest.

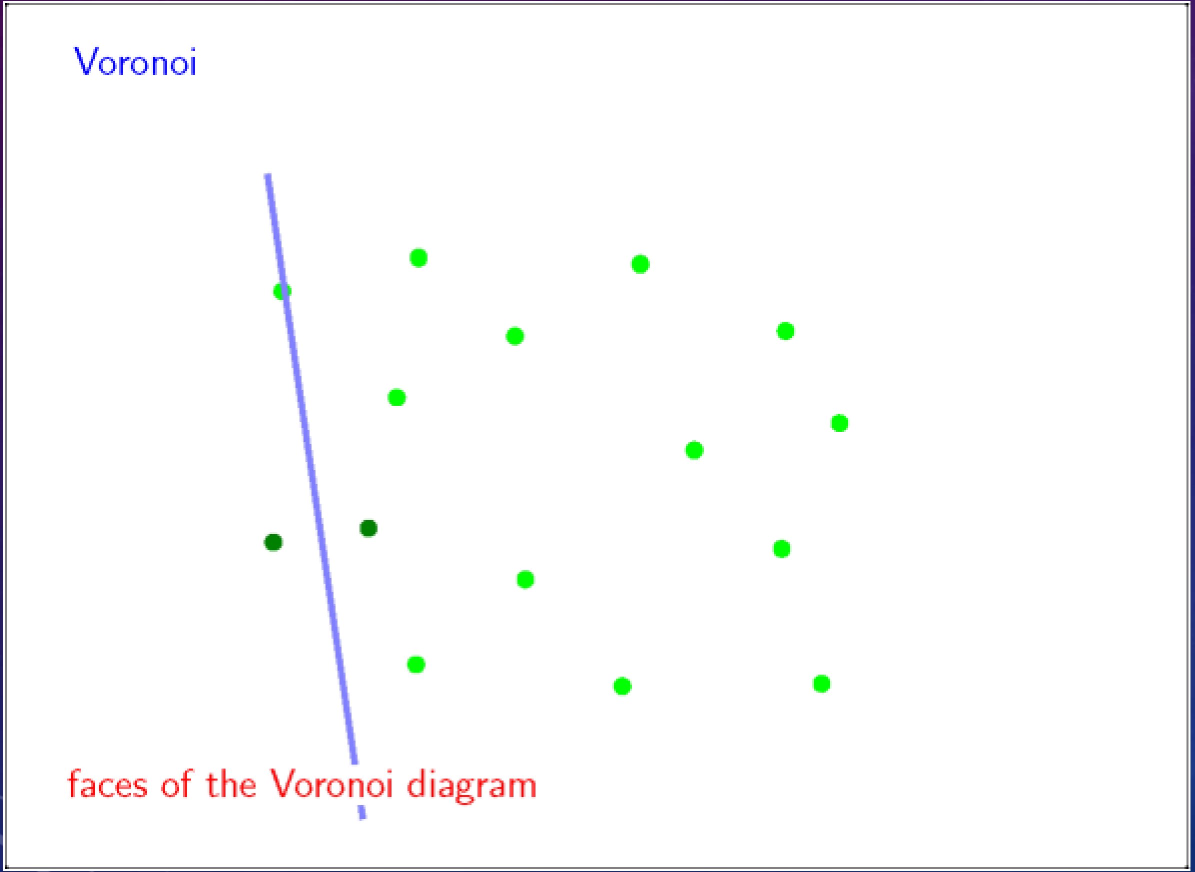


The bisector of two points.

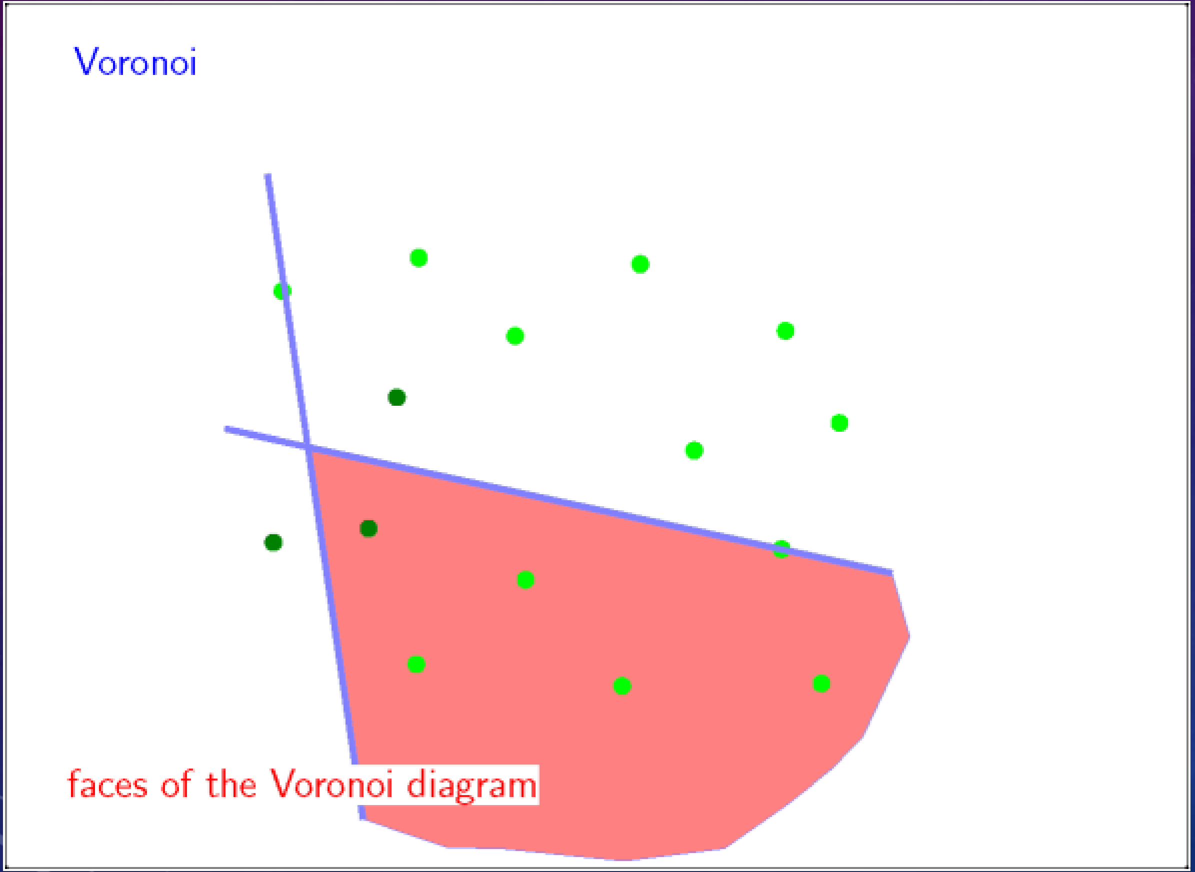
Post Office Problem



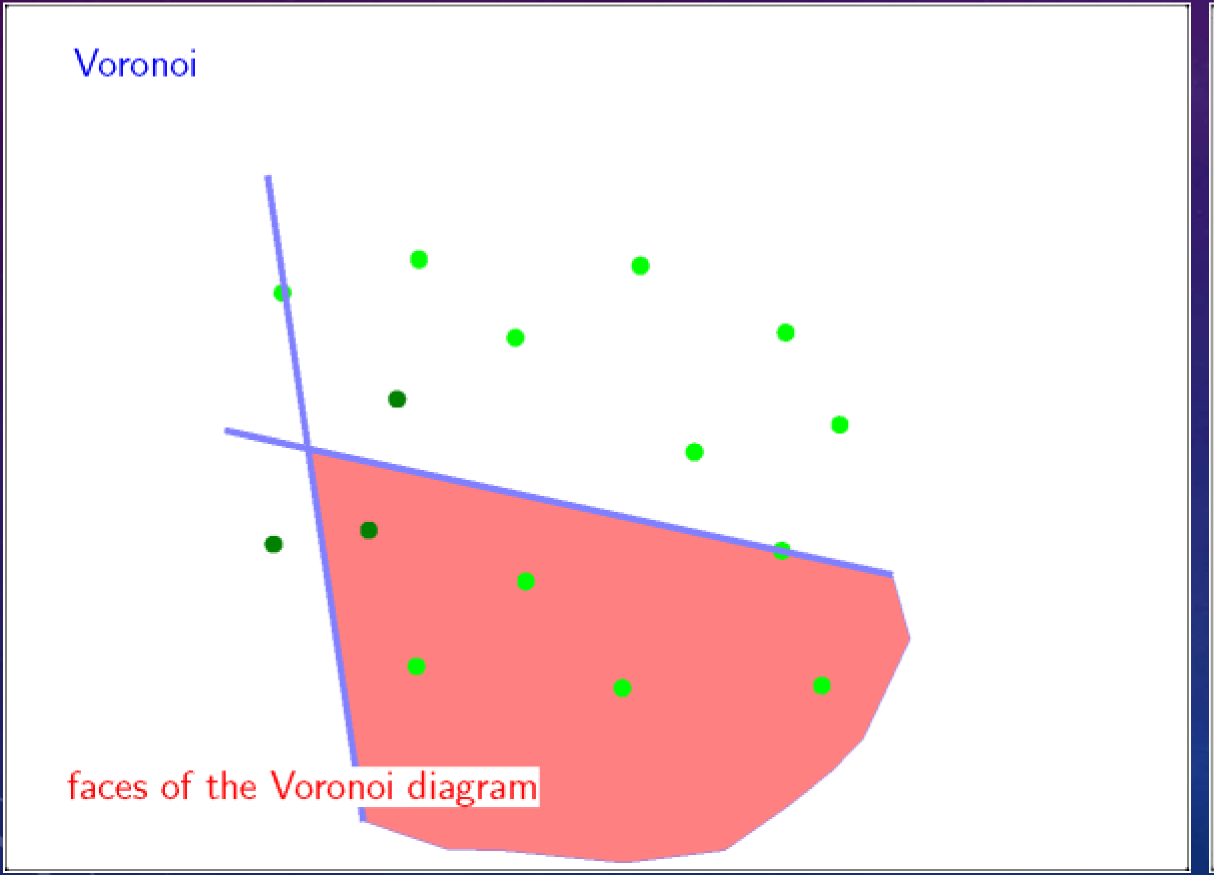
Post Office Problem

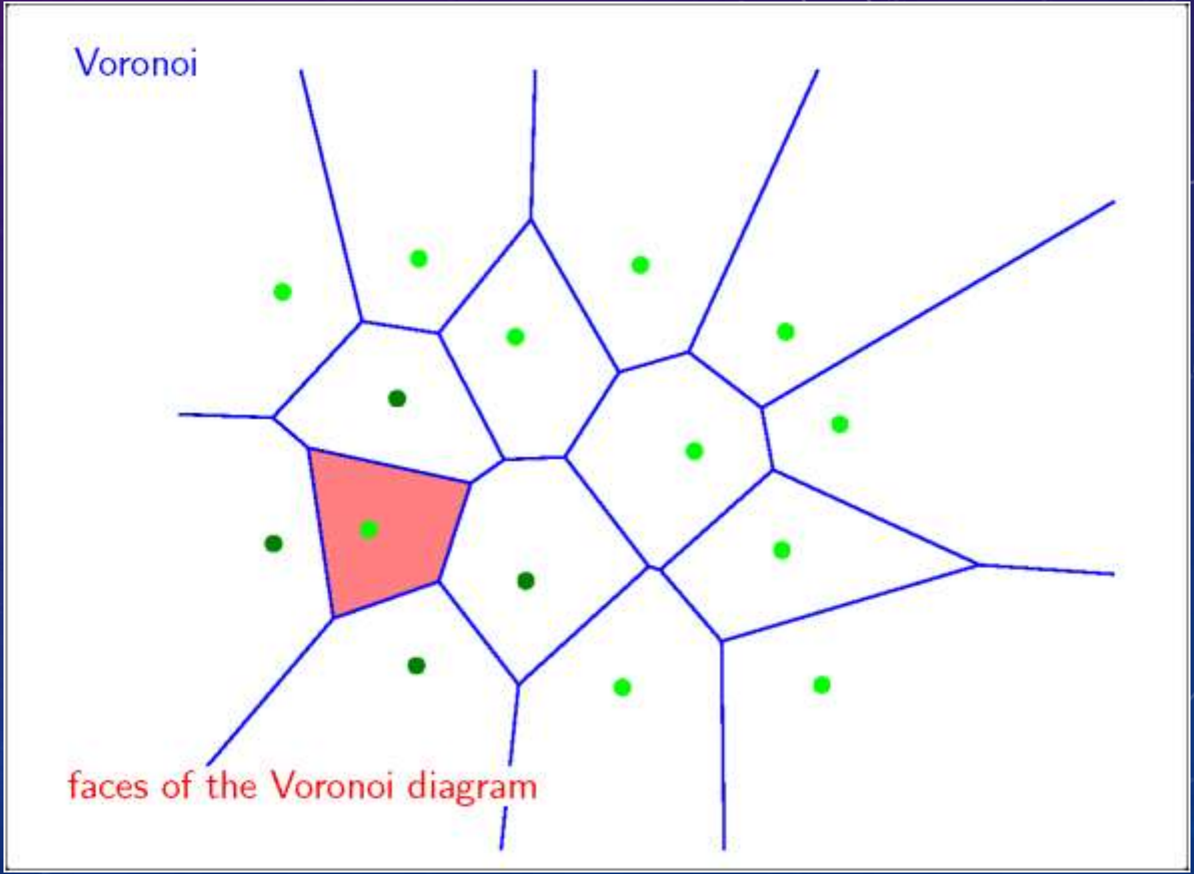


Post Office Problem



Post Office Problem





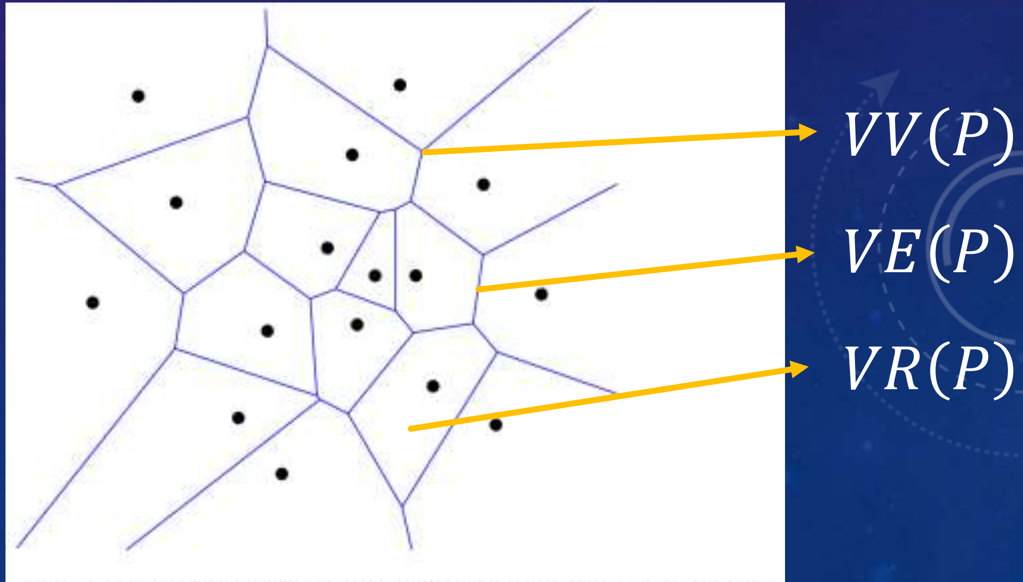
Voronoi cell

- Given a set $P = \{p_1, ..., p_n\}$ of points in $\mathbb{R}^2$, for $p_i \in P$ denote the Voronoi cell $VP(i)$ of p_i by

$$VP(i) \triangleq \{q \in \mathbb{R}^2, ||q - p_i|| \leq ||q - p||, \forall p \in P\}$$

Property:

- $VP(i) = \cap_{j \neq i} H(p_i, p_j)$
- $VP(i)$ is non-empty and convex.
- $VP(i)$ form a subdivision of the plane.



Example: The Voronoi diagram of a point set.

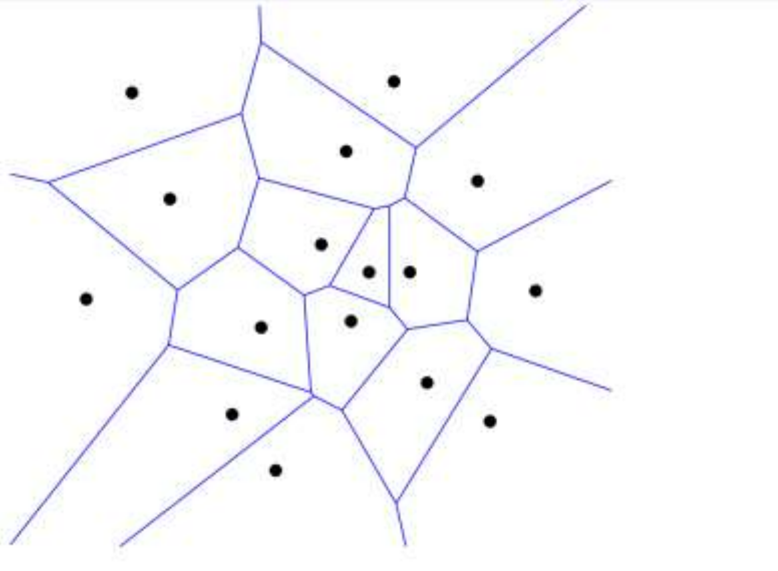
Lemma 1

- For every vertex $v \in VV(P)$ the following statements hold.

1) v is the common intersection of at least three edges from $VE(P)$;

2) v is incident to at least three regions from $VR(P)$;

Proof:
As all Voronoi cells are convex, each interior angle is less than π, thus $k \geq 3$ of them must be incident to v.



Example: The Voronoi diagram of a point set.

Lemma 1

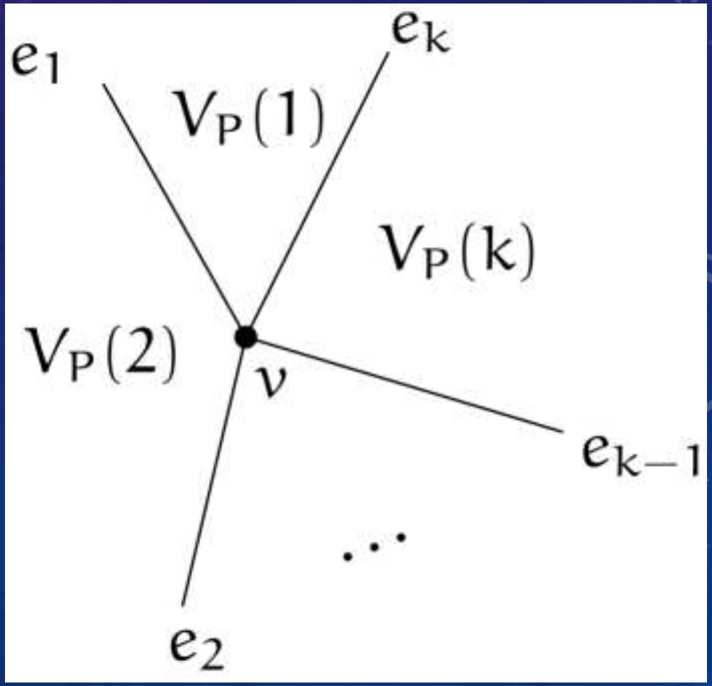
- For every vertex $v \in VV(P)$ the following statements hold.

1) v is the common intersection of at least three edges from $VE(P)$;

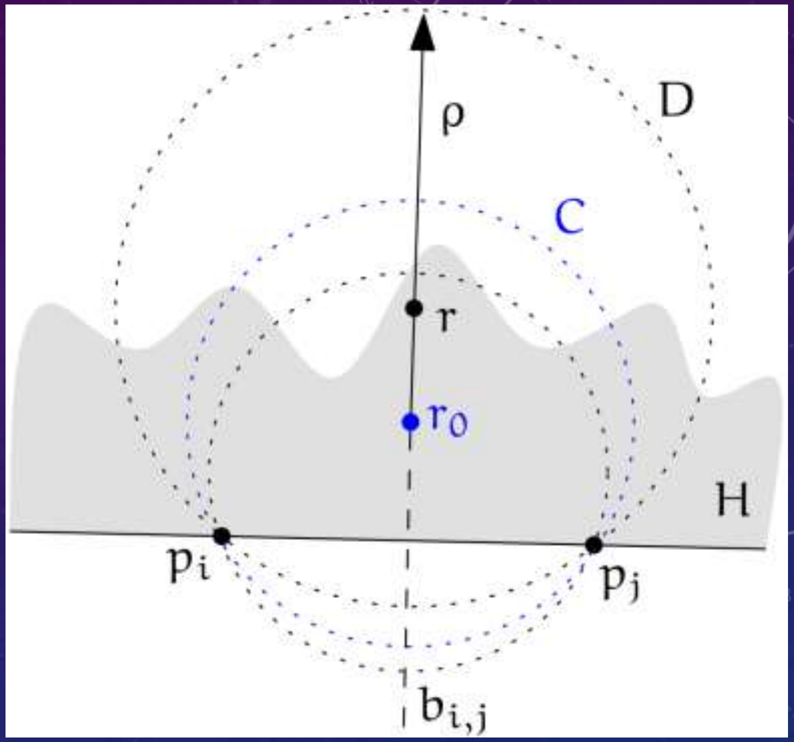
2) v is incident to at least three regions from $VR(P)$;

3) v is the center of a circle $C(v)$ through at least three points from P and $C(v)^{\circ} \cap P = \emptyset$;

Suppose there exists a point $p_l \in C(v)^{\circ}$. Then the vertex v is closer to p_l than it is to any of $p_1, \ldots, p_k$, in contradiction to $v \in VP(i), i = 1, \ldots, k$.



Lemma 2



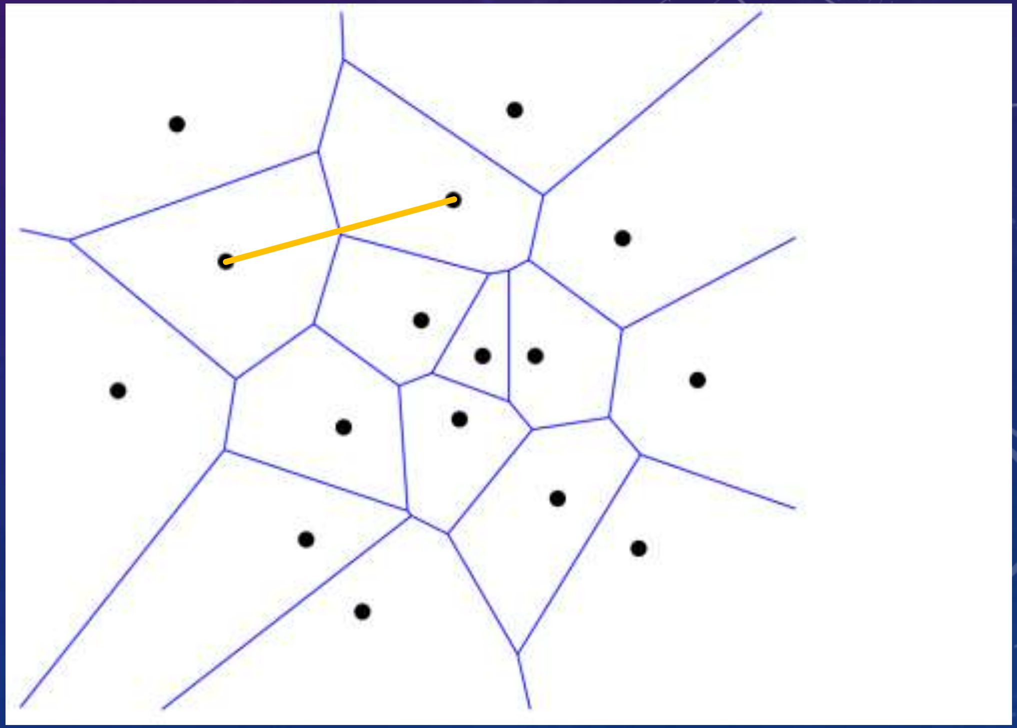
- There is an unbounded Voronoi edge bounding $VP(i)$ and $VP(j) \Leftrightarrow \overline{p_i p_j} \cap P = \{p_i, p_j\}$ and $\overline{p_i p_j} \in \partial conv(P)$ where the latter denotes the boundary of the convex hull of P.

Proof: There is an unbounded Voronoi edge bounding $VP(i)$ and $VP(j) \Leftrightarrow$ there is a ray $\rho \subset b_{i,j}$ such that $\|r - p_k\| > \|r - p_i\| (= \|r - p_j\|), \forall r \in \rho$ and $p_k \in P \backslash \{p_i, p_j\}$. Equivalently, there is a ray $\rho \subset b_{i,j}$ such that for every point $r \in \rho$ the circle $C \in D$ centered at r does not contain any point from P in its interior.

Duality

- A straight-line dual of a plane graph G is a graph G' defined as follows:

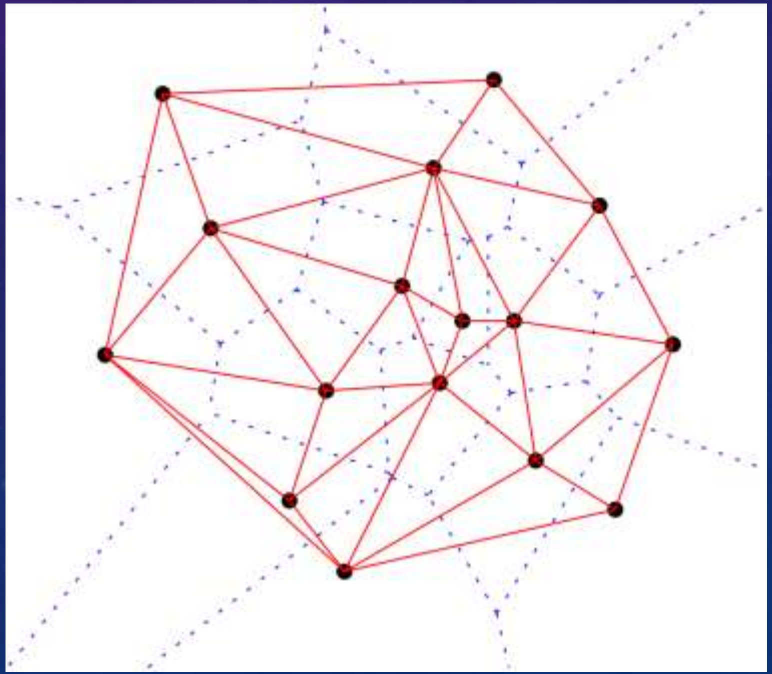
choose a point for each face of G and connect any two such points by a straight edge, if the corresponding faces share an edge of G



Delaunay triangulation

- Theorem: The straight-line dual of $VD(P)$ for a set $P \subset \mathbb{R}^2$ of $n > 3$ points in general position (no three points from P are collinear and no four points from P are cocircular) is a triangulation: the unique Delaunay triangulation of P.

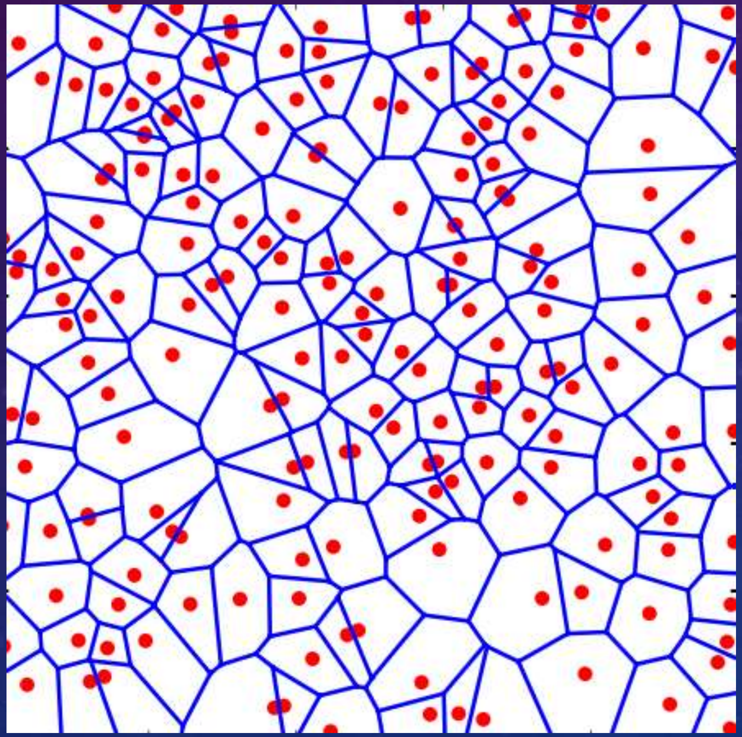
Proof: $\Longrightarrow$
1. convex hull
2. Triangles
3. Empty circle property



Proof: $\Longleftarrow$
1. Circumcenter is selected for each face.
2. Empty circle property.

Centroidal Voronoi tessellations (CVT)

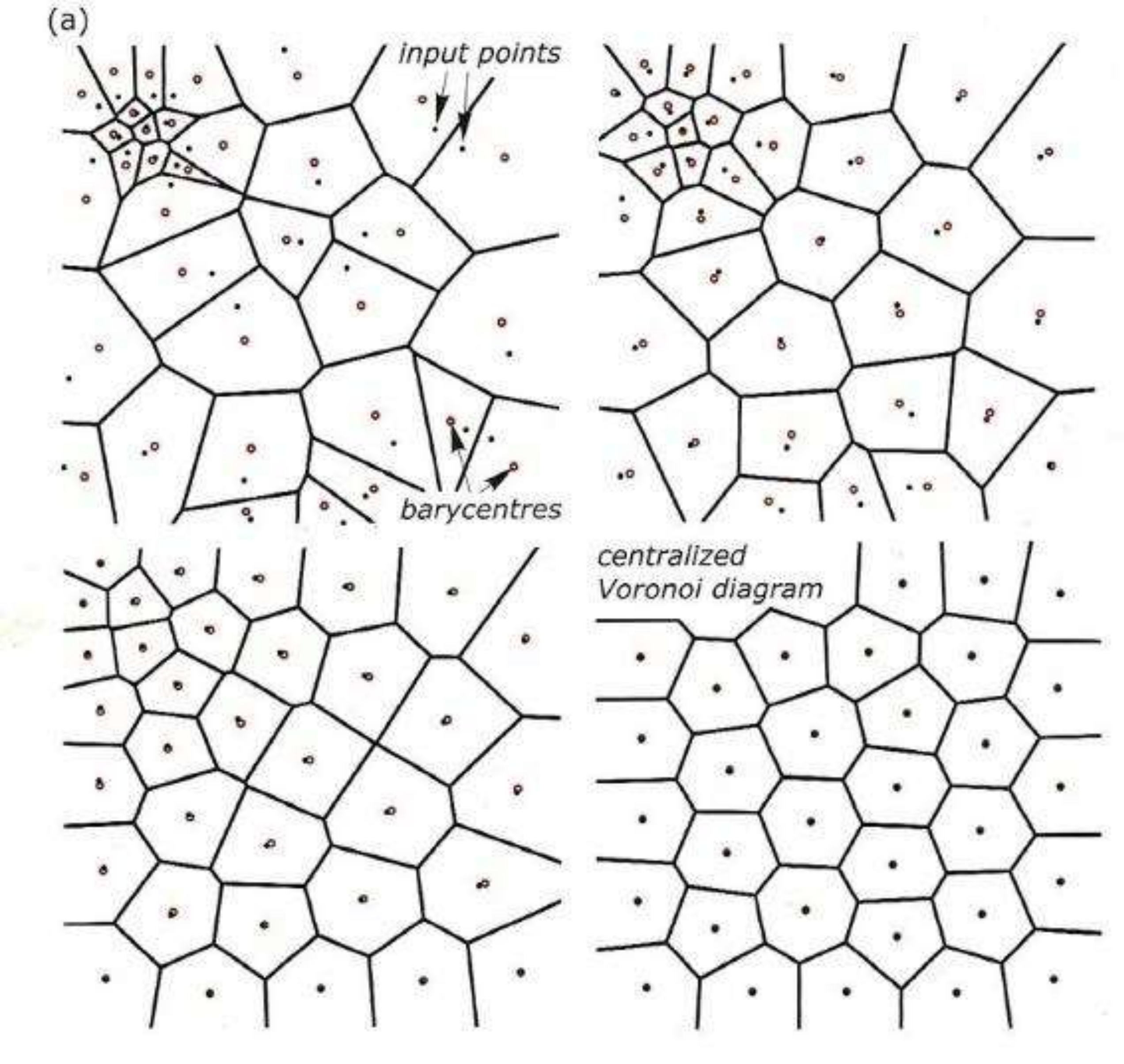
- Update vertices



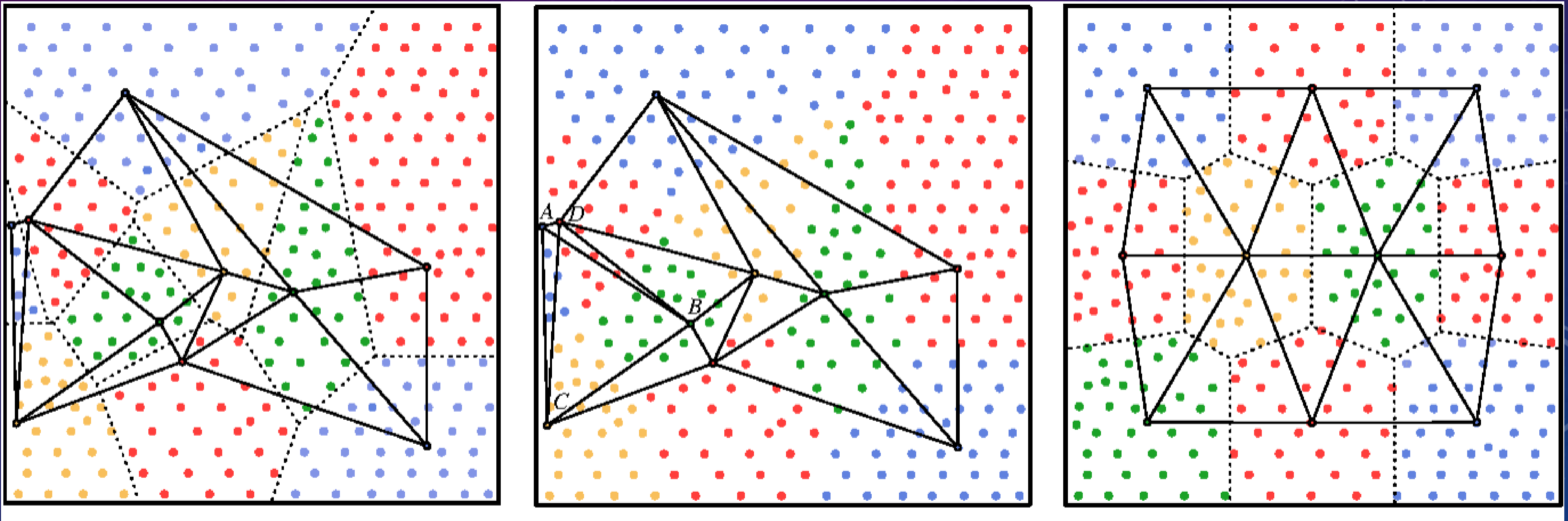
Definition – CVT

- A class of Voronoi tessellations where each site coincides with the centroid (i.e., center of mass) of its Voronoi region.

$$c_i = \frac{\int_{V_i} x\rho(x)dx}{\int_{V_i} \rho(x)dx}$$



Applications – Remeshing



(a) (b) (c)

Energy function

$$E(p_1, \dots, p_n, V_1, \dots, V_n) = \sum_{i=1}^{n} \int_{V_i} \|x - p_i\|^2 dx$$

- For a fixed set of sites $P = \{p_1, \dots, p_n\}$, the energy function is minimized if $\{V_1, \dots, V_n\}$ is a Voronoi tessellation.
- For the fixed regions, the p_i are the mass centroids c_i of their corresponding regions V_i.

Lloyd iteration

- Construct the Voronoi tessellation corresponding to the sites p_i.
- Compute the centroids c_i of of the Voronoi regions V_i and move the sites p_i to their respective centroids c_i.
- Repeat above steps until satisfactory convergence is achieved.

Lloyd iteration